MATH 1132

Solutions to Practice Exam 2

1(A)

If the n^{th} partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = 1 + \frac{n}{3^n}$

then $a_n = \frac{2-n}{3^n}$ for $n > 1$ (a) T F

Solution: __________ FALSE ...

1. $s_n = a_1 + a_2 + a_3 + \ldots a_{n-1} + a_n = s_{n-1} + a_n$
2. $s_n = s_{n-1} + a_n \quad \Longrightarrow \quad a_n = s_n - s_{n-1}$
3. $a_n = (1 + \frac{n}{3^n}) - (1 + \frac{n-1}{3^{n-1}}) = \frac{n}{3^n} - \frac{n-1}{3^{n-1}}$
4. $a_n = \frac{n}{3^n} - \frac{3(n-1)}{3^n} = \frac{n-3(n-1)}{3^n} = \frac{3-2n}{3^n} \neq \frac{2-n}{3^n}$

1(B)

The geometric series $\sum_{n=4}^{\infty}(\frac{1}{3})^n$ converges to $\frac{3}{2}$. (b) T F

______________________________________ FALSE

Solution 1:

1. $s_n = (\frac{1}{3})^4 + (\frac{1}{3})^5 + \ldots + (\frac{1}{3})^n$
2. $(\frac{1}{3})s_n = \qquad (\frac{1}{3})^5 + \ldots + (\frac{1}{3})^n + (\frac{1}{3})^{n+1} \ldots$
3. $(\frac{2}{3})s_n = (\frac{1}{3})^4 - (\frac{1}{3})^{n+1} \implies s_n = (\frac{3}{2})\frac{1}{3^4} - (\frac{3}{2})(\frac{1}{3})^{n+1}$
4. $\lim_{n\to\infty} s_n = (\frac{3}{2})\frac{1}{3^4} - (\frac{3}{2})\lim_{n\to\infty}(\frac{1}{3})^{n+1} = \frac{1}{2\cdot 3^3} = \frac{1}{54}$

Solution 2:

If $|r| < 1$ then $\displaystyle\sum_{k=0}^{\infty} a\, r^k = \frac{a}{1-r}$. If $|r| \geq 1$ the series diverges.

1. $$\sum_{n=0}^{\infty} (\frac{1}{3})^n = 1 + \frac{1}{3} + (\frac{1}{3})^2 + (\frac{1}{3})^3 + \sum_{n=4}^{\infty} (\frac{1}{3})^n$$
2. $$\frac{1}{2/3} = \frac{40}{27} + \sum_{n=4}^{\infty} (\frac{1}{3})^n$$ $r = 1/3$
3. $$\sum_{n=4}^{\infty} (\frac{1}{3})^n = \frac{3}{2} - \frac{40}{27} = \frac{1}{54}$$

1(c)

If $\lim_{n\to\infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} a_n$ converges. (c) T F

Solution: __________ FALSE

A counter example

1. $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series ($p = 1$)
2. $\lim_{n\to\infty} \frac{1}{n} = 0$

1(d)

The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$ converges conditionally. (d) T F

Solution: ___________ FALSE

1. $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k^3} \right| = \sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series $(p = 3 > 1)$
2. The given series is absolutely convergent, not conditionally convergent.

1(E)

If $\sum_{n=1}^{\infty} |a_n|$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges. __________ FALSE

1. $\sum_{n=1}^{\infty} |(-1)^n \frac{1}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series $(p = 1)$
2. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is a convergent series. The answer is: FALSE

What is true:

- If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} |a_n|$ diverges.

1(F)

The sequence $a_n = \dfrac{\ln(2n)}{\ln(n)}$ converges to 1. (f) T F

Solution: __________ TRUE ...

1. $$\lim_{n\to\infty} \frac{\ln(2n)}{\ln(n)} = \lim_{n\to\infty} \frac{\ln(2) + \ln(n)}{\ln(n)}$$

2. $$\lim_{n\to\infty} \frac{\dfrac{\ln 2}{\ln n} + 1}{1} = 1$$

Or, use L'Hospital's Rule on $\dfrac{\ln(2x)}{\ln(x)}$

1(g)

If the power series $\sum_{k=0}^{\infty} a_k (x-4)^k$ has a radius of convergence equal to 2 then $\sum_{k=0}^{\infty} a_k$ diverges. (g) T F

Solution: __________ FALSE ...

1. The center is $x = 4$ so the interval of convergence is one of the following:
   - $(2,6),\ [2,6],\ [2,6),\ (2,6]$ All include $x = 5$
2. When $x = 5$ the series looks like $\sum_{k=0}^{\infty} a_k (x-4)^k = \sum_{k=0}^{\infty} a_k$
3. When $x = 5$ the series converges since 5 is inside all these intervals

2(A)

Which of the following sequences is both bounded and monotonic?

(i) $a_n = n^2$ (ii) $a_n = \dfrac{n}{n+1}$ (iii) $a_n = \dfrac{\sin(\pi n)}{n}$ (iv) $a_n = \dfrac{n}{\sqrt{n+1}}$

(i) $a_n = n^2$ not bounded: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty$

(ii) $a_n = \dfrac{n}{n+1}$ is bounded and monotonic

- Bounded: $0 < \dfrac{n}{n+1} < \dfrac{n+1}{n+1} = 1$ bounded
- Monotonic:
  - $f(x) = \dfrac{x}{x+1} \implies f'(x) = \dfrac{x+1-x}{(x+1)^2} = \dfrac{1}{(x+1)^2} > 0$
  - f increasing $\implies a_n = f(n) = \dfrac{n}{n+1}$ is increasing

(iii) $a_n = \dfrac{\sin(\pi n)}{n}$ bounded and monotonic: Every term is 0

(iv) $a_n = \dfrac{n}{\sqrt{n+1}}$: Monotonic but not bounded

- $a_n = \dfrac{\sqrt{n^2}}{\sqrt{n+1}} = \sqrt{\dfrac{n^2}{n+1}} = \sqrt{\dfrac{n}{1+1/n}} \longrightarrow \infty$ not bounded

- $f(x) = \dfrac{x}{(x+1)^{1/2}} \Longrightarrow f'(x) = \dfrac{(x+1)^{1/2} - (x/2)(x+1)^{-1/2}}{x+1}$

  $f'(x) = \dfrac{x+1-(x/2)}{(x+1)^{3/2}} = \dfrac{1+(x/2)}{(x+1)^{3/2}} > 0$

  $f(x)$ is increasing, hence $f(n) = a_n$ is monotonic

2(B): $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)$

$$S_4=\sum_{k=1}^{4}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)$$

$$S_4=\frac{1}{1}+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}-\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)-\frac{1}{5}=1-\frac{1}{5}$$

$$S_n=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$S_n=1+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}-\frac{1}{3}\right)+\ldots+\left(-\frac{1}{n}+\frac{1}{n}\right)-\frac{1}{n+1}$$

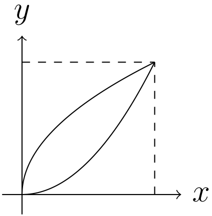
$$S_n=1-\frac{1}{n+1}$$

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$$

3.

$$f(x) = \sqrt{x}$$

$$g(x) = x^2$$



$$\bar{x} = \frac{M_y}{\text{mass}} = \frac{\rho \displaystyle\int_a^b x(f(x) - g(x))\,dx}{\rho \displaystyle\int_a^b (f(x) - g(x))\,dx}$$

$$\bar{y} = \frac{M_x}{\text{mass}} = \frac{\rho \displaystyle\int_a^b \frac{1}{2}[(f(x))^2 - (g(x))^2]\,dx}{\rho \displaystyle\int_a^b (f(x) - g(x))\,dx}$$

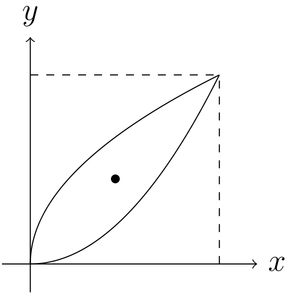
$$\text{Area } = \int_0^1 \sqrt{x} - x^2\,dx = \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{3}$$

$$\int_0^1 x(\sqrt{x} - x^2)\,dx = \int_0^1 x^{3/2} - x^3\,dx = \left(\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right)\Big|_0^1 = \frac{3}{20}$$

$$\frac{1}{2}\int_0^1 (\sqrt{x})^2 - (x^2)^2\,dx = \frac{1}{2}\int_0^1 x - x^4\,dx = \frac{1}{2}\,(\frac{1}{2}x^2 - \frac{1}{5}x^5)\Big|_0^1 = \frac{3}{20}$$

$$\overline{x} = \frac{3/20}{1/3} = \frac{9}{20} \quad \text{and} \quad \overline{y} = \frac{3/20}{1/3} = \frac{9}{20}$$

The centroid is $(\frac{9}{20}, \frac{9}{20})$.



4. Consider the following series, all of which converge. For which of these series do you get a conclusive answer when using the **Ratio Test** to check for convergence? Write the letters of all possible answers. If no series satisfies this condition, write "none". You do not need to show your work.

A $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{2k^5 + k^2 + 1}$ **B** $\displaystyle\sum_{k=1}^{\infty} \frac{k^6}{k!}$ **C** $\displaystyle\sum_{k=1}^{\infty} (3k+4)^{-k}$

D $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ **E** $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{2}{5^k}$

To use the Ratio Test on $\displaystyle\sum_{k=1}^{\infty} a_k$ we compute the limit

$$r = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}| \qquad \text{(Must have all positive terms)}$$

4A $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{2k^5+k^2+1}$ $\quad r = 1 \quad$ **ratio test gives no conclusion**

$$\lim_{k\to\infty} \frac{\dfrac{(k+1)^3}{2(k+1)^5+(k+1)^2+1}}{\dfrac{k^3}{2k^5+k^2+1}} = \lim_{k\to\infty} \frac{(k+1)^3(2k^5+k^2+1)}{k^3(2(k+1)^5+(k+1)^2+1)}$$

$$= \lim_{k\to\infty} \frac{(k+1)^3}{k^3} \quad \lim_{n\to\infty} \frac{2k^5+k^2+1}{2(k+1)^5+(k+1)^2+1}$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^3 \quad \lim_{k\to\infty} \frac{2+1/k^3+1/k^5}{2(k+1)^5/k^5+(k+1)^2/k^5+1/k^5} = 1$$

4B $\displaystyle\sum_{k=1}^{\infty} \frac{k^6}{k!}$ $\quad r = 0 < 1$ **Converges by the ratio test**

$$r = \lim_{k\to\infty} \frac{\dfrac{(k+1)^6}{(k+1)!}}{\dfrac{k^6}{k!}} = \lim_{k\to\infty} \frac{(k+1)^6 k!}{k^6 (k+1) k!} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^6 \frac{1}{k+1} = 0$$

4C $\displaystyle\sum_{k=1}^{\infty} (3k+4)^{-k}$ $\quad r = 0 < 1$ **Converges by the ratio test**

$$r = \lim_{k\to\infty} \frac{(3(k+1)+4)^{-(k+1)}}{(3k+4)^{-k}} = \lim_{k\to\infty} \frac{(3k+4)^k}{(3k+7)^{(k+1)}}$$

$$= \lim_{k\to\infty} \frac{(3k+4)^k}{(3k+7)^k} \frac{1}{(3k+7)} = \lim_{k\to\infty} \frac{1}{(3k+7)} = 0 < 1$$

4D $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ $\quad r = 1$ **No conclusion from ratio test**

$$\lim_{k\to\infty} \frac{\dfrac{\ln(k+1)}{(k+1)^2}}{\dfrac{\ln k}{k^2}} = \lim_{k\to\infty} \frac{k^2 \ln(k+1)}{(k+1)^2 \ln k} = \lim_{k\to\infty} \frac{k^2}{(k^2+2k+1)} \frac{\ln(k+1)}{\ln k}$$

$$= \lim_{k\to\infty} \frac{1}{1+2/k+1/k^2} \frac{\ln(k+1)}{\ln k} = \lim_{k\to\infty} \frac{\ln(k+1)}{\ln k}$$

$$\lim_{k\to\infty} \frac{\ln(k+1)}{\ln k} = \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} = \lim_{x\to\infty} \frac{\dfrac{1}{x+1}}{\dfrac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} = 1$$

$r = 1$ No conclusion

4E $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{2}{5^k}$ **Absolutely convergent**

Apply ratio test to $\displaystyle\sum_{k=1}^{\infty}|(-1)^k\frac{2}{5^k}| = \sum_{k=1}^{\infty}\frac{2}{5^k}$

- $$r = \lim_{k\to\infty}\frac{\frac{2}{5^{(k+1)}}}{\frac{2}{5^k}} = \lim_{k\to\infty}\frac{2\cdot 5^k}{2\cdot 5^{(k+1)}} = \lim_{k\to\infty}\frac{1}{5} < 1 = \frac{1}{5}$$

5(i) $a_n = \left(\dfrac{1-2n}{n+1}\right)^2$ bounded?, increasing?, convergent?

$$a_n = \frac{1-4n+4n^2}{n^2+2n+1} = \frac{1/n^2+4/n+4}{1+2/n+1/n^2} \longrightarrow 4 \quad \text{as} \quad n \to \infty$$

This sequence is convergent, hence also bounded.

To see if it is increasing, we consider $f(x) = \left(\dfrac{1-2x}{x+1}\right)^2$ ...

$$f'(x) = 2\left(\frac{1-2x}{x+1}\right)\frac{-3}{(x+1)^2} = 2\left(\frac{2x-1}{x+1}\right)\left(\frac{3}{(x+1)^2}\right)$$

$f'(x) > 0$ for $x \geq 1$

$f(x)$ is increasing hence $f(k) = a_k$ is also increasing

5 (ii), (iii)

(ii) $b_n = 3^{n+5}2^{-n}$

$$3^{n+5}2^{-n} = \frac{3^{n+5}}{2^n} = 3^5\ \frac{3^n}{2^n} = 3^5\ \left(\frac{3}{2}\right)^n$$

This is an unbounded, divergent geometric sequence that is increasing (each term is $\dfrac{3}{2}$ times the previous term)

(iii) $c_n = \dfrac{(-5)^{n+1}}{(3)^n}$

$$\frac{(-5)^{n+1}}{(3)^n} = (-1)^{n+1}5\frac{5^n}{3^n} = (-1)^{n+1}5\ \left(\frac{5}{3}\right)^n$$

An unbounded, divergent geometric sequence oscillating between positive and negative values. It is not increasing.

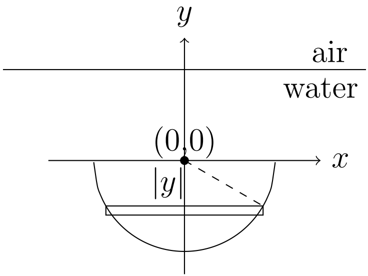
5(iv) $S_n = \sum_{k=2}^{n} \frac{k}{k^3-2}$

$S_{n+1} = S_n + \frac{n+1}{(n+1)^3-2} \implies S_n$ is increasing

- Use Limit Comparison Test on $\sum_{k=2}^{\infty} \frac{k}{k^3-2}$
- $\frac{k}{k^3-2} \sim \frac{1}{k^2} \Rightarrow$ Compare to the convergent p-series $\sum_{k=2}^{\infty} \frac{1}{k^2}$
- $\lim_{k\to\infty} \frac{\frac{1}{k^2}}{\frac{k}{k^3-2}} = \lim_{k\to\infty} \frac{k^3-2}{k^3} = \lim_{k\to\infty} (1 - \frac{2}{k^3}) = 1$

$\lim_{n\to\infty} S_n = \sum_{k=2}^{\infty} \frac{k}{k^3-2}$ converges by the Limit Comparison Test and hence is also bounded.

6



The top lies 1 meter below the surface. Let the origin $(0,0)$ be the top center of the plate. The radius is $1/2$, so the lower boundary of the plate is $y = -\sqrt{1/4 - x^2}$. The depth of the y-slice is $1 - y$.

The pressure at that depth is $P(y) = 1000g(1-y) = 9800(1-y)$

Since $x^2 + y^2 = (\frac{1}{2})^2$, $2x = 2\sqrt{1/4 - y^2}$ is the width of the y-slice

The area of the y-slice is $A(y) \approx 2\sqrt{1/4 - y^2}\,\Delta y$ and the hydrostatic force on the y-slice is given by

$$F(y) \approx P(y)A(y) = 9800(1-y)(2\sqrt{1/4 - y^2}\,\Delta y.$$

$$\text{Hydrostatic force on the plate} = \int_{-1/2}^{0} 9800(1-y)\sqrt{1/4 - y^2}\,dy.$$

7a: $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k^2+1}}{k}$ converges or diverges?

$$\frac{\sqrt{k^2+1}}{k} \approx \frac{\sqrt{k^2}}{k} = \frac{k}{k} = 1 \text{ for large } k$$

Consider the Divergence Test: $\displaystyle\lim_{k\to\infty} \frac{\sqrt{k^2+1}}{k} = \lim_{k\to\infty} \frac{\sqrt{k^2+1}}{\sqrt{k^2}}$

$$= \lim_{k\to\infty} \sqrt{\frac{k^2+1}{k^2}} = \lim_{k\to\infty} \sqrt{\frac{1+1/k^2}{1}} = 1$$

Diverges by the Divergence Test

7B: $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k\ln k}$ converges or diverges?

Use Integral Test with $f(x)=\dfrac{1}{x\ln x}$. f is continuous, positive

$$f(x)=\frac{1}{x\ln x}\Longrightarrow f'(x)=\frac{-(\ln x+1)}{(x\ln x)^2}<0. \qquad f(x) \text{ is decreasing}$$

Let $u=\ln x$. $\displaystyle\int\frac{1}{x\ln x}\,dx=\int\frac{1}{u}\,du=\ln|u|+C=\ln|\ln x|+C$

$$\int_2^{\infty}\frac{1}{x\ln x}\,dx=\lim_{b\to\infty}\int_2^{b}\frac{1}{x\ln x}\,dx=\lim_{b\to\infty}\ln|\ln b|-\ln|\ln 2|=\infty$$

Conclusion: $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k\ln k}$ diverges by the Integral Test

7C: $\displaystyle\sum_{k=2}^{\infty} ke^{-2k^2} = \sum_{k=2}^{\infty} \frac{k}{e^{2k^2}}$

The Integral Test works but the Ratio Test is simpler

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\dfrac{k+1}{e^{2(k+1)^2}}}{\dfrac{k}{e^{2k^2}}} = \lim_{k\to\infty} \frac{(k+1)\ e^{2k^2}}{k\ e^{2k^2+4k+2}}$$

$$= \lim_{k\to\infty} (1+\frac{1}{k})(\frac{1}{e^{4k+2}}) = 0 < 1$$

Conclusion: $\displaystyle\sum_{k=2}^{\infty} ke^{-2k^2}$ converges by the Ratio Test

7D: $\sum_{k=0}^{\infty} \frac{4+3^k}{4^k}$ Two choices:

- Limit Comparison: $\frac{4+3^k}{4^k} \approx (\frac{3}{4})^k$

  Compare to the convergent geometric series $\sum_{k=0}^{\infty} (\frac{3}{4})^k$

- Ratio Test:

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{4+3^{k+1}}{4^{k+1}}}{\frac{4+3^k}{4^k}} = \lim_{k\to\infty} \frac{4^k(4+3^{k+1})}{4^{k+1}(4+3^k)} = \frac{3}{4} < 1$$

Conclusion: $\sum_{k=0}^{\infty} \frac{4+3^k}{4^k}$ converges by the Ratio Test

7E: $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$ INTEGRAL TEST

$f(x) = \frac{1}{x(\ln x)^3}$ is a positive, decreasing, continuous function

Consider $\int \frac{1}{x(\ln x)^3}\,dx$ and take $u = \ln x$

$$\int \frac{1}{x(\ln x)^3}\,dx = \int \frac{1}{u^3}\,du = \frac{-1}{2u^2} + C = \frac{-1}{2(\ln x)^2} + C$$

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty} \left(\frac{1}{2(\ln 2)^2} - \frac{1}{2(\ln b)^2} \right)$$

Conclusion: $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$ converges by the Integral Test

7F: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k^2+1}}$ Test for absolute convergence first

- Use Limit Comparison Test on $\displaystyle\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}}{\sqrt{k^2+1}}\right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$
- $\displaystyle\frac{1}{\sqrt{k^2+1}} \sim \frac{1}{\sqrt{k^2}} = \frac{1}{k} \Rightarrow$ Compare to the divergent $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$
- $$\lim_{k\to\infty} \frac{\frac{1}{k}}{\frac{1}{\sqrt{k^2+1}}} = \lim_{k\to\infty} \frac{\sqrt{k^2+1}}{k} = \lim_{k\to\infty} \frac{\sqrt{k^2+1}}{\sqrt{k^2}}$$
$$= \lim_{k\to\infty} \sqrt{\frac{k^2+1}{k^2}} = \lim_{k\to\infty} \sqrt{\frac{1+1/k^2}{1}} = 1$$
- $\displaystyle\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}}{\sqrt{k^2+1}}\right|$ diverges by Limit Comparison Test

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{\sqrt{k^2+1}} \right|$$ diverges by Limit Comparison Test

Now apply the **Alternating Series Test** to $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k^2+1}}$

1. The terms $(-1)^k \dfrac{1}{\sqrt{k^2+1}}$ alternate sign
2. $\displaystyle\lim_{k\to\infty} \frac{1}{\sqrt{k^2+1}} = 0.$
3. Let $f(x) = \dfrac{1}{\sqrt{x^2+1}} = (x^2+1)^{-1/2}$.

   $f'(x) = -\dfrac{1}{2}(x^2+1)^{-3/2}(2x) = -x(x^2+1)^{-3/2} < 0.$

   f is decreasing $\Longrightarrow a_k = f(k) = \dfrac{1}{\sqrt{k^2+1}}$ decreasing

Conclusion: This series converges by the Alternating Series Test and thus is conditionally convergent.

7G: $\displaystyle\sum_{k=1}^{\infty} \frac{k^4}{e^{3k}}$

- Ratio Test:

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\dfrac{(k+1)^4}{e^{3(k+1)}}}{\dfrac{k^4}{e^{3k}}} = \lim_{k\to\infty} \frac{(k+1)^4}{k^4}\frac{e^{3k}}{e^{3k+3}}$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^4 \frac{1}{e^3} = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^4 \frac{1}{e^3} = \frac{1}{e^3} < 1$$

Conclusion: $\displaystyle\sum_{k=1}^{\infty} \frac{k^4}{e^{3k}}$ converges by the Ratio Test

7H: $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k^2-1}$

- Use Limit Comparison Test on $\displaystyle\sum_{k=2}^{\infty} \left|\frac{(-1)^k}{k^2-1}\right| = \sum_{k=2}^{\infty} \frac{1}{k^2-1}$
- $\displaystyle\frac{1}{k^2-1} \sim \frac{1}{k^2} \Rightarrow$ Compare to the convergent p-series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2}$
- $\displaystyle\lim_{k\to\infty} \frac{\frac{1}{k^2}}{\frac{1}{k^2-1}} = \lim_{k\to\infty} \frac{k^2-1}{k^2} = \lim_{k\to\infty} \frac{1-1/k^2}{1} = 1$

$\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k^2-1}$ converges absolutely by Limit Comparison Test

8: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k2^k}(x-2)^k$: Find the Interval of convergence

Step 1. Ratio Test on $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k2^k}|x-2|^k$

- $$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\dfrac{1}{(k+1)2^{k+1}}|x-2|^{k+1}}{\dfrac{1}{k2^k}|x-2|^k}$$

  $$= \lim_{k\to\infty} \frac{k\ 2^k}{(k+1)2^{k+1}}|x-2| = \lim_{k\to\infty} \frac{k}{k+1}\frac{|x-2|}{2} = \frac{|x-2|}{2}$$

- Absolute convergence when $\dfrac{|x-2|}{2} < 1$ or when $|x-2| < 2$

Absolute convergence when

$$|x-2| < 2 \Longrightarrow -2 < x-2 < 2 \Longrightarrow 0 < x < 4$$

Step 2. Test $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k2^k}(x-2)^k$ for convergence at $x = 0$ and 4

- Take $x = 0$: $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k2^k}(-2)^k = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k}$
  - Converges by the Alternating Series Test
- Take $x = 4$: $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k2^k}(2)^k = \sum_{k=0}^{\infty} \frac{1}{k}$
  - Divergent p-series $(p = 1)$

- The **interval of convergence** is $[0, 4)$
- **Center** $x = 2$
- $R = 2$ is the **radius of convergence**

9: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n} x^n$: Interval of convergence

Step 1. Ratio Test on $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k} |x|^k$

- $$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{3^{k+1}}{k+1}|x|^{k+1}}{\frac{3^k}{k}|x|^k}$$

  $$= \lim_{k\to\infty} \frac{3^{k+1}}{3^k} \frac{k}{k+1} |x| = \lim_{k\to\infty} \frac{1}{1+1/k} 3|x| = 3|x|$$

- Absolute convergence when $3|x| < 1$ or when $|x| < \dfrac{1}{3}$

Absolute convergence when $|x| < \frac{1}{3} \Longrightarrow -\frac{1}{3} < x < \frac{1}{3}$

Step 2. Test $\sum_{n=1}^{\infty} \frac{3^k}{k} x^k$ for convergence at $x = \pm\frac{1}{3}$

- Take $x = -\frac{1}{3}$: $\quad \sum_{k=1}^{\infty} \frac{3^k}{k}(-\frac{1}{3})^k = \sum_{k=1}^{\infty}(-1)^k \frac{1}{k}$
  - Converges by the Alternating Series Test
- Take $x = \frac{1}{3}$: $\quad \sum_{k=1}^{\infty} \frac{3^k}{k}(\frac{1}{3})^k = \sum_{k=1}^{\infty} \frac{1}{k}$ Divergent p-series

- The **interval of convergence** is $[-\frac{1}{3}, \frac{1}{3})$
- $R = \frac{1}{3}$ is the **radius of convergence**

10. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{2^n}(x-1)^n$ has a radius of convergence $R=2$.

(a) Explain carefully what the "radius of convergence" tells us about the series.

- The radius of convergence $R=2$ gives the radius of the interval of convergence centered at $x=1$. The series converges whenever $|x-1|<2$. It may or may not converge when $|x-1|=2$ (at the endpoints of the interval of convergence). The series diverges if $|x-1|>2$.

(b) Find the interval of convergence.

- Possible answers: $(-1,3)$, $(-1,3]$, $[-1,3)$, $[-1,3]$
- We need to test the endpoints, when $x=-1$ and $x=3$.

$\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{2^n}(x-1)^n$: Testing endpoints

1. Let $x=-1$ $\quad \sum_{n=1}^{\infty}(-1)^n\frac{n^2}{2^n}(-2)^n$
   - $\sum_{n=1}^{\infty}(-1)^n(-1)^n\frac{n^2}{2^n}(2)^n = \sum_{n=1}^{\infty}n^2$
   - Diverges by the divergence test since $\lim_{n\to\infty} n^2 \neq 0$
2. Let $x=3$ $\quad \sum_{n=1}^{\infty}(-1)^n\frac{n^2}{2^n}(2)^n$
   - $\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{2^n}(2)^n = \sum_{n=1}^{\infty}(-1)^n n^2$
   - Diverges by the divergence test since $\lim_{n\to\infty}(-1)^n n^2 \neq 0$
3. Interval of convergence: $(-1,3)$

11. How many terms of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{7n+5}$ do we need to add in order to approximate the series with $|\text{error}| < 0.00001$)?

The approximation $\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{7k+5}$ has an error less than the absolute value of the first neglected term, namely $\frac{1}{7(n+1)+5}$.

We want an n that makes $\frac{1}{7(n+1)+5} < 0.00001 = \frac{1}{10^5}$

$$7n + 12 > 10^5 \quad \Longrightarrow \quad n > \frac{10^5 - 12}{7} = 14284 \text{ will suffice}$$

We need 14, 284 terms.

$$\sum_{k=1}^{14{,}284}(-1)^{k+1}\frac{1}{7k+5} \approx \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{7k+5} \text{ has error} < 0.00001$$