Important Notice: To prepare for the final exam, one should study the past exams and practice midterms (and homeworks, quizzes, and worksheets), not just this practice final. A topic not being on the practice final does not mean it won't appear on the final.

1. If the statement is always true, circle the printed capital T. If the statement is sometimes false, circle the printed capital F. In each case, write a careful and clear justification or a counterexample.

   (a) If a force of $F(x) = 6x$ pounds is required to stretch a spring x feet beyond its rest length, then 36 ft-lbs of work is done in stretching the spring from its natural length to 6 feet beyond its rest length. False

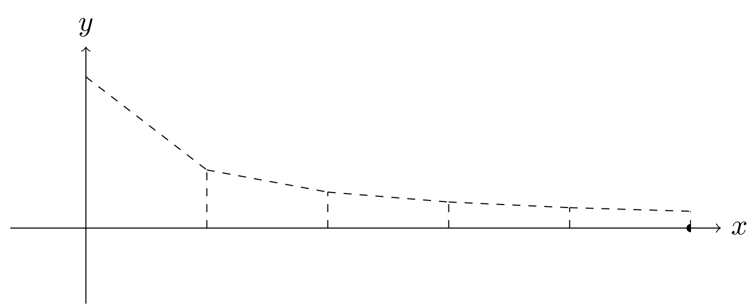
   Work: $W = \int_a^b F(x)\,dx$

   Work done stretching the spring from $x = 0$ to $x = 6$:

   $$W = \int_0^6 6x\,dx = 3x^2\Big|_0^6 = 108 \text{ ft-lbs}$$

   (b) The trapezoid rule with $n = 5$ for $\int_0^4 \frac{dx}{2x+1}$ will be an overestimate. True

   Justification: The curve $y = 1/(2x + 1)$ is decreasing and concave up, so the trapezoids cover the graph throughout. The estimate is larger than the integral.



   (c) $\ln(2.5) = 1.5 - \frac{1}{2}(1.5)^2 + \frac{1}{3}(1.5)^3 - \frac{1}{4}(1.5)^4 + \frac{1}{5}(1.5)^5 - \ldots$ False

   We have
   $$\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots$$
   for $-1 < x \leq 1$; the series on the right diverges for $|x| > 1$. Therefore when we set $x = 1.5$, the left side is $\ln(2.5)$ but the series on the right becomes divergent.

(d) The improper integral $\displaystyle\int_1^\infty \frac{x^2}{(x^3+7)^{1/3}}\,dx$ converges. False

$$\begin{aligned}\int_1^\infty \frac{x^2}{(x^3+7)^{1/3}}\,dx &= \lim_{b\to\infty}\int_1^b \frac{x^2}{(x^3+7)^{1/3}}\,dx\\ &= \lim_{b\to\infty} \frac{1}{2}(x^3+7)^{2/3}\Big|_1^b\\ &= \lim_{b\to\infty}\left(\frac{1}{2}(b^3+7)^{2/3}-\frac{1}{2}(8)^{2/3}\right)=\infty\end{aligned}$$

This limit does not exist so $\displaystyle\int_1^\infty \frac{x^2}{(x^3+7)^{1/3}}\,dx$ does not converge,

(e) The tangent line to the parametric curve $(x,y)=(t-1/t, 4+t^2)$ at the point where $t=1$ has equation $y=x+5$. True

Slope $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{2t}{1+1/t^2}$

$\dfrac{dy}{dx}=\dfrac{2}{2}=1$ when $t=1$

$y=x+b$ and passes through $(x,y)=(0,5)$

$y=x+5$

2. For each multiple choice question, circle the correct answer. There is only one correct choice for each answer.

(a) A cylindrical tank with a radius of 1 meter and a height of 8 meters is half full. Letting $y=0$ correspond to the top of the tank and ρ be the density of water, the work required to pump the water out of the tank is

(a) $\pi\rho g\displaystyle\int_4^8 y\,dy$ (b) $\pi\rho g\displaystyle\int_0^8 y\,dy$ (c) $\pi\rho g\displaystyle\int_0^4 y\,dy$ (d) $16\pi\rho g\displaystyle\int_4^8 y\,dy$

The y-values for the water in the tank are $4\le y\le 8$. Partition $[4,8]$ into k parts.

$V_k=\pi r^2\Delta y=\pi\Delta y$ weight $=\pi\Delta y\cdot\rho\cdot g$

k-th slice is lifted y_k meters (since $y=0$ is the top of the tank).

$W_k\approx\rho g\pi\, y_k\Delta y$ $\displaystyle\sum_{k=1}^n W_k\approx\sum_{k=1}^n\rho g\pi\, y_k\,\Delta y$

$W=\pi\rho g\displaystyle\int_4^8 y\,dy$, which is (a). It is also $\pi\rho g\displaystyle\int_0^4(8-y)\,dy$.

($\rho=1000$ kg/m^3 & $g=9.8\,$m/s^2)

(b) The Taylor series at $x = 0$ for $\sin x$ is

(a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{n!}$ (b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$ (c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!}$ (d) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$

Taylor Series at $x = a$ is $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$.

Let $f(x) = \sin x$ and $a = 0$. Then

- $f(0) = f^{(4)}(0) = \sin 0 = 0$
- $f'(0) = f^{(5)}(0) = \cos 0 = 1$
- $f''(0) = f^{(6)}(0) = -\sin 0 = 0$
- $f^{(3)}(0) = f^{(7)}(0) - \cos 0 = -1$

The Taylor series for $\sin x$ at 0 is $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$.

(c) A parametric curve tracing out the circle once clockwise for $0 \le t \le \pi$ starting at $(1, 0)$ is

(a) $(\cos t, \sin t)$ traces out the top half of the circle counter-clockwise starting at $(1, 0)$.

(b) $(\cos t, -\sin t)$ traces out the bottom half of the circle clockwise starting at $(1, 0)$.

(c) $(\cos(2t), \sin(2t))$ traces out the complete circle counter-clockwise starting at $(1, 0)$.

(d) $(\cos(2t), -\sin(2t))$ traces out the complete circle clockwise starting at $(1, 0)$.

The answer is (d).

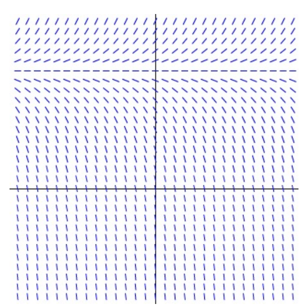
(d) Which differential equation has the direction field shown?

(i) $y'(t) = 6 - 3y$

(ii) $y'(t) = y$

(iii) $y'(t) = 3y - 6$

(iv) $y'(t) = (y - 2)e^t$



Answer: (iii) $y'(t) = 3y - 6$. Slope depends only on y, is negative for small positive y, and is positive for larger y.

3. Let R be the region enclosed by the curves $y = 2x$ and $y = x^2$. Write a definite integral that gives the volume of the solid generated by rotating the region R around the line $y = 6$.

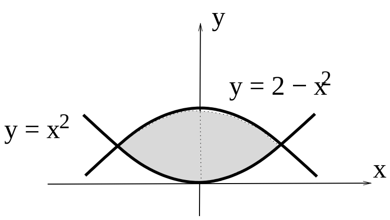
   Curves meet when $2x = x^2 \Rightarrow x^2 - 2x = x(x-2) = 0$

   $x = 0, 2 \Longrightarrow (0,0), (2,4)$ are the intersection points, so R lies over $0 \le x \le 2$.

   Partition $[0,2]$ along the x-axis. The area of the x-slice is $A(x) = \pi R^2 - \pi r^2$ where $R = 6 - x^2$ and $r = 6 - 2x$.

   $$V = \int_0^2 A(x)\,dx = \pi \int_0^2 \left((6-x^2)^2 - (6-2x)^2\right)\,dx$$

4. A solid has a base bounded by the curves $y = x^2$ and $y = 2 - x^2$ for $-1 \le x \le 1$. Cross-sections perpendicular to the x-axis are squares. Write a definite integral for its volume.



   - Set $x^2 = 2 - x^2$
   - $2 - 2x^2 = 0 \Longrightarrow x = \pm 1$
   - Base curves cross when $x = -1$ and $x = 1$

   Side of square in the x-slice is $s = (2 - x^2) - x^2 = (2 - 2x^2)$
   Area of cross-section of x-slice is $A(x) = s^2 = (2-2x^2)^2$

   By the slicing method: $V = \displaystyle\int_a^b A(x)\,dx = \int_{-1}^1 (2-2x^2)^2\,dx.$

5. Compute: (a) $\displaystyle\int x\cos x\,dx$ using integration by parts, (b) $\displaystyle\int \frac{x+1}{x(x-4)}\,dx$ using partial fractions.

   (a) We use $\displaystyle\int u\,dv = uv - \int v\,du.$

   Choose $u = x$ and $dv = \cos x\,dx \Longrightarrow du = dx$ and $v = \sin x$. Then

   $$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

(b) $\dfrac{x+1}{x(x-4)} = \dfrac{A}{x} + \dfrac{B}{(x-4)} = \dfrac{A(x-4)}{x(x-4)} + \dfrac{Bx}{x(x-4)} = \dfrac{A(x-4)+Bx}{x(x-4)}$.

Clearing the denominator, $x+1 = A(x-4) + Bx$.

Solving for A, B:

- Set $x = 0 \Longrightarrow 1 = -4A \Longrightarrow A = -\dfrac{1}{4}$
- Set $x = 4 \Longrightarrow 5 = 4B \Longrightarrow B = \dfrac{5}{4}$

$$\int \frac{x+1}{x(x-4)}\,dx = \int \left(\frac{A}{x} + \frac{B}{x-4}\right) dx = -\frac{1}{4}\int \frac{dx}{x} + \frac{5}{4}\int \frac{dx}{x-4} \ = -\frac{1}{4}\ln|x| + \frac{5}{4}\ln|x-4| + C.$$

6. Use the error bound formulas on the last page to determine an n such that the trapezoid rule with n subintervals approximates $\displaystyle\int_0^1 \frac{1}{e^x}\,dx$ to within .001.

$|error| \le \dfrac{K(b-a)^3}{12n^2}$, where K fits $|f''(x)| \le K$ on $[0,1]$

$f(x) = e^{-x}, \quad f'(x) = -e^{-x}, \quad f''(x) = e^{-x}$

Since $f''(x)$ is decreasing, on $[0,1]$ we have $|f''(x)| = e^{-x} \le f''(0) = 1$, so we can use $K = 1$. Thus

$$|error| \le \frac{K(b-a)^3}{12n^2} = \frac{1(1^3)}{12n^2} \overset{?}{\le} 0.001 = \frac{1}{1000}$$

Solve for n: $n^2 \ge \dfrac{1000}{12} = 83.333$ The least n we can choose by this method is $n = 10$.

7. (a) Obtain the Taylor series for $\dfrac{1}{1+x}$ at $x = 0$ from the geometric series for $\dfrac{1}{1-x}$.

- $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \displaystyle\sum_{k=0}^{\infty} x^k$ for $|x| < 1$
- Replacing x with $-x$, $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \displaystyle\sum_{k=0}^{\infty} (-1)^k x^k$ for $|x| < 1$

(b) Use your result from part (a) and integration to write down the Taylor series at $x = 0$ for $\ln(1+x)$ and then find its interval of convergence.

Integrating termwise, $\displaystyle\int \frac{1}{1+x}\,dx = \int 1\,dx - \int x\,dx + \int x^2\,dx - \int x^3\,dx + \ldots$, so

$$\ln(1+x) + C = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}.$$

To find C, let $x = 0$: $\ln(1) + C = 0 \Longrightarrow C = 0$. Thus

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} x^{k+1} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}.$$

The radius of convergence comes from the ratio test: when $a_k = (-1)^{k-1}/k$, $|a_{k+1}/a_k| = k/(k+1) \to 1$ as $k \to \infty$, so the radius of convergence is 1. At the endpoints ± 1, the series converges when $x = 1$ (alternating harmonic series) and diverges when $x = -1$ (negative of harmonic series), so the interval of convergence is $(-1, 1]$.

8. Find the 3rd-order Taylor polynomial centered at 4 for $\dfrac{1}{\sqrt{x}}$.

$$T_3(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3$$

- $f(x) = \dfrac{1}{\sqrt{x}} = x^{-1/2}$
- $f'(x) = -\dfrac{1}{2}x^{-3/2} = -\dfrac{1}{2x^{3/2}}$
- $f''(x) = \dfrac{3}{4}x^{-5/2} = \dfrac{3}{4x^{5/2}}$
- $f'''(x) = -\dfrac{15}{8}x^{-7/2} = -\dfrac{15}{8x^{7/2}}$
- $f(4) = \dfrac{1}{2}$
- $f'(4) = -\dfrac{1}{2(2^3)} = -\dfrac{1}{16}$
- $f''(4) = \dfrac{3}{4(2^5)} = \dfrac{3}{128}$
- $f'''(4) = -\dfrac{15}{8(2^7)} = -\dfrac{15}{1024}$

Thus

$$T_3(x) = \frac{1}{2} - \frac{1}{16}(x-4) + \frac{3}{128(2!)}(x-4)^2 - \frac{15}{1024(3!)}(x-4)^3.$$

9. How many terms of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ do we need to add to estimate the series with $|\text{error}| < 0.001$?

The series is alternating, so the error in approximating the series using the nth partial sum is at most the magnitude of the first omitted term. Therefore if we use the nth partial sum, the error is $< \dfrac{1}{(n+1)^2+1}$. We seek n such that $\dfrac{1}{(n+1)^2+1} < 0.001 = \dfrac{1}{1000}$.

$(n+1)^2 + 1 > 1000 \iff (n+1)^2 > 999 \iff n+1 > 31.6.$ We need $n > 30.6$, so $n \geq 31$. Thus $\displaystyle\sum_{n=0}^{31} \frac{(-1)^n}{n^2+1}$, the sum of the first 32 terms (from $n = 0$ to $n = 31$) approximates the full series with the desired error.

10. Use the integral test to show $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $0 < p < 1$.

Let $f(x) = \dfrac{1}{x^p} = x^{-p}$. Then

- $f'(x) = -px^{-p-1} = -\dfrac{p}{x^{p+1}} < 0$ so $f(x)$ decreases,
- $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x^p} = 0.$

Suppose $p > 1$. Then $1 - p < 0$

- $\displaystyle\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{b\to\infty} \int_1^b x^{-p}\, dx = \lim_{b\to\infty} \frac{x^{1-p}}{1-p}\bigg|_0^b = \lim_{b\to\infty} \frac{b^{1-p}-1}{1-p} = \frac{-1}{1-p} = \frac{1}{p-1}.$
- Thus $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$.

Suppose $0 < p < 1$. Then $1 - p > 0$

- $\displaystyle\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{b\to\infty} \int_1^b x^{-p}\, dx = \lim_{b\to\infty} \frac{x^{1-p}}{1-p}\bigg|_0^b = \lim_{b\to\infty} \frac{b^{1-p}-1}{1-p} = \infty.$
- So $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges for $0 < p < 1$.

11. Determine which of the following series converges conditionally, converges absolutely or diverges. Specify which convergence test you use and show how it leads to the answer.

(a) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ Test for absolute convergence first.

Compare $\displaystyle\sum_{k=1}^{\infty} \left|(-1)^k \frac{1}{\ln k}\right| = \sum_{k=1}^{\infty} \frac{1}{\ln k}$ to $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ using the Comparison Test.

$\ln k < k \implies \dfrac{1}{\ln k} > \dfrac{1}{k}$ To show $\ln k < k$ for $k \geq 2$, we look at

- $f(x) = x - \ln x$. Then $f'(x) = 1 - 1/x > 0$ for $x > 1$, so $f(x)$ increasing for $x > 1$. Thus $x \geq 2 \implies f(x) \geq f(2) = 2 - \ln 2 > 0$, so $x > \ln x$ for $x \geq 2$.

$\displaystyle\sum_{k=2}^{\infty} \frac{1}{k}$ diverges $\implies \displaystyle\sum_{k=2}^{\infty} \frac{1}{\ln k}$ diverges (by the Comparison Test)

Now use the Alternating Series Test since the series is not absolutely convergent

Set $b_k = 1/\ln k$. Two ways to show b_k is decreasing: (1) the denominators $\ln k$ are increasing or (2) consider $f(x) = 1/\ln x$ for $x > 1$ and its first derivative:

$$f'(x) = -\frac{1}{(\ln x)^2} \frac{1}{x} < 0.$$

That is negative, so $f(x)$ is decreasing for $x > 1$.

Conclusion: The series converges by the Alternating Series Test but is not absolutely convergent. It is conditionally convergent.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{k^2 + 50}$

1. $\displaystyle\lim_{k\to\infty} \frac{k^2}{k^2 + 50} = \lim_{k\to\infty} \frac{1}{1 + 50/k^2} = 1 \neq 0$

2. The series diverges by the Divergence Test

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k^5}{k!}$ Apply the Ratio Test to test for absolute convergence.

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\dfrac{(k+1)^5}{(k+1)!}}{\dfrac{k^5}{k!}} = \frac{(k+1)^5}{k^5}\frac{k!}{(k+1)!} = \left(\frac{k+1}{k}\right)^5 \frac{1}{k+1} = \left(1+\frac{1}{k}\right)^5 \frac{1}{k+1}$$

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^5 \frac{1}{k+1} = 0 < 1.$$

Thus $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k^5}{k!}$ converges absolutely.

(d) $\displaystyle\sum_{k=0}^{\infty} \frac{5}{2^k+5k+3}$ Apply Ratio Test (terms all positive, so no need for absolute values)

$$\frac{a_{k+1}}{a_k} = \frac{\dfrac{5}{2^{k+1}+5(k+1)+8}}{\dfrac{5}{2^k+5k+3}} = \frac{2^k+5k+3}{2^{k+1}+5(k+1)+8} = \frac{1+5k/2^k+3/2^k}{2+5(k+1)/2^k+8/2^k}$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{1}{2} < 1.$$

By the Ratio Test, the series $\displaystyle\sum_{k=1}^{\infty} \frac{5}{2^k+5k+3}$ converges absolutely.

12. Solve for y exactly:

(a) $\dfrac{dy}{dx} = \dfrac{\sin x}{y^2}$ with $y(0) = 3$.

1. Write as $y^2\,dy = \sin x\,dx$
2. $\displaystyle\int y^2\,dy = \int \sin x\,dx$
3. $\dfrac{y^3}{3} = -\cos x + C$
4. Apply initial condition $y(0) = 3$: $9 = -1 + C \Longrightarrow C = 10$.
5. Thus $\dfrac{y^3}{3} = 10 - \cos x \Longrightarrow y = (30 - 3\cos x)^{1/3}$.

(b) $\dfrac{dy}{dx} = y\cos x + xy$ with $y(0) = 3$.

1. $\dfrac{dy}{dx} = y(\cos x + x) \Longrightarrow \dfrac{1}{y}\,dy = (\cos x + x)\,dx$

2. $\displaystyle\int \frac{1}{y}\,dy = \int (\cos x + x)\,dx$

3. $\ln|y| = \sin x + \dfrac{x^2}{2} + C \Longrightarrow |y| = e^{\sin x + x^2/2 + C} = e^C e^{\sin x + x^2/2}$

4. $y = \pm e^C e^{\sin x + x^2/2} = Ke^{\sin x + x^2/2}$, where $K = \pm e^C$.

Apply initial condition $y(0) = 3$: $3 = Ke^{\sin 0 + 0^2/2} = K$

Solution is $y = 3e^{\sin x + x^2/2}$.

13. Find the orthogonal trajectories of the family of curves $y = kx^4$.

For points (x, y) on the curve $y = kx^4$, we have $k = y/x^4$ and

$$\frac{dy}{dx} = 4kx^3 = 4\frac{y}{x^4}x^3 = \frac{4y}{x}.$$

A curve orthogonal to this family must have derivative (slope) equal to the negative reciprocal of $4y/x$, hence an orthogonal curve to the family has

$$\frac{dy}{dx} = \frac{-x}{4y}.$$

- $4y\,dy = -x\,dx$
- $\displaystyle\int 4y\,dy = -\int x\,dx$
- $2y^2 = -\frac{1}{2}x^2 + C$ for any C.

Thus the families $y = kx^4$ and $\frac{1}{2}x^2 + 2y^2 = C$ are **orthogonal trajectories** of each other.

14. A tank contains 60 L of water with 5 kg of salt dissolved in it. Brine that contains 2 kg of salt per liter enters the tank at a rate of 3 L/min. Pure water is also flowing into the tank at a rate of 2 L/min. The solution in the tank is kept well mixed and is drained at a rate of 5 L/min. How much salt remains in the tank after 30 minutes? What happens in the long run?

Let $y(t)$ = amount of salt (in kg) after t minutes, so $y(0) = 5$ kg.

Then dy/dt = inflow rate of salt − outflow rate of salt.

The volume remains constant since the rate of flow in is the same as the rate of the flow out, namely 5 L/min, so the concentration (in kg/L) of salt in the tank after t minutes is $y(t)/60$.

- inflow rate $= 2\dfrac{\text{kg}}{L} \times 3\dfrac{\text{L}}{\text{min}} + 0\dfrac{\text{kg}}{L} \times 2\dfrac{\text{L}}{\text{min}} = 6\dfrac{\text{kg}}{\text{min}}$.
- outflow rate $= \dfrac{y(t)}{60}\dfrac{\text{kg}}{L} \times 5\dfrac{\text{L}}{\text{min}} = \dfrac{y(t)}{12}\dfrac{\text{kg}}{\text{min}}$

Our differential equation becomes $\dfrac{dy}{dt} = 6 - \dfrac{y(t)}{12}, \quad y(0) = 5.$

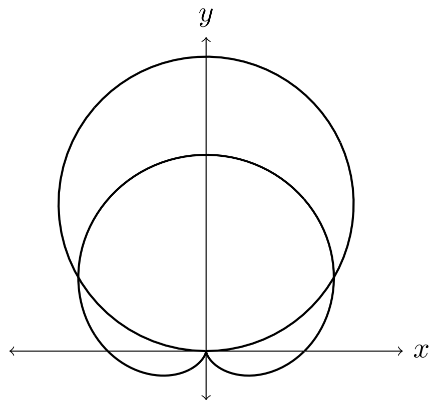
$$\frac{dy}{dt} = \frac{72 - y}{12} \implies \frac{dy}{72 - y} = \frac{1}{12}\, dt$$

- $\displaystyle\int \frac{1}{72 - y}\, dy = \int \frac{1}{12}\, dt$
- $-\ln|72 - y| = \dfrac{t}{12} + C \implies \ln|72 - y| = -\dfrac{t}{12} - C \implies |72 - y| = e^{-C}e^{-t/12}$
- $72 - y = \pm e^{-C}e^{-t/12} = Ke^{-t/12} \implies y(t) = 72 - Ke^{-t/12}$

From the initial condition $y(0) = 5$ we have $5 = 72 - K$, so $K = 72 - 5 = 67$. Thus the solution to the differential equation is $y(t) = 72 - 67e^{-t/12}$.

- $y(30) = 72 - 67e^{-30/12} \approx 66.5$ kg
- The long-run behavior is found from $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} (72 - 67e^{-t/12}) = 72$ kg.

15. Below are graphs of $r = 3\sin\theta$ and $r = 1 + \sin\theta$.



(a) Determine both polar and rectangular coordinates for all points where the curves cross.
Finding crossing points:

$$3\sin\theta = 1 + \sin\theta \iff 2\sin\theta = 1 \iff \sin\theta = \frac{1}{2}.$$

Thus $\theta = \pi/6$ and $5\pi/6$ at crossing points, and here $r = 3\sin\theta = 3/2$.
Polar coordinates of crossing points: $(r,\theta) = (3/2, \pi/6),\ (3/2, 5\pi/6)$
Rectangular coordinates: $(x, y) = (r\cos\theta, r\sin\theta) = (3\sqrt{3}/4, 3/4),\ (-3\sqrt{3}/4, 3/4)$

(b) Set up, but do **not** evaluate, an integral for the area of the region inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$.

The rays $\pi/6 \le \theta \le 5\pi/6$ "sweep out" the region.

To see which curve corresponds to which equation, write $r = 3\sin\theta$ in rectangular coordinates: $r^2 = 3r\sin\theta$ is the same as $x^2 + y^2 = 3y$, so $x^2 + (y^2 - 3y + 9) = 9$, so $x^2 + (y - 3)^2 = 3^2$. That is a circle centered at $(0, 3)$ with radius 3. The other curve is $r = 1 + \sin\theta$.

The area inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$ is

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left((3\sin\theta)^2 - (1 + \sin\theta)^2\right)\, d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left((3\sin\theta)^2 - (1 + \sin\theta)^2\right)\, d\theta.$$

Error Bound Formulas

Trapezoid Rule and Error Bound: Let $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ with $x_i - x_{i-1} = \Delta x = \dfrac{b-a}{n}$ for all i. Then the nth approximation T_n to $\int_a^b f(x)\,dx$ using the trapezoid rule is

$$T_n = (f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n))\frac{\Delta x}{2}$$

and

$$\left|\int_a^b f(x)\,dx - T_n\right| \leq \frac{K(b-a)}{12}(\Delta x)^2 = \frac{K(b-a)^3}{12n^2},$$

where K is an upper bound on $|f''(x)|$ over $[a,b]$: $|f''(x)| \leq K$ for $a \leq x \leq b$.

Taylor's Inequality:

Let $T_n(x)$ be the nth-order Taylor polynomial for $f(x)$ at $x = a$ and $|f^{(n+1)}(c)| \leq M$ for all c between a and x. Then

$$|T_n(x) - f(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}$$