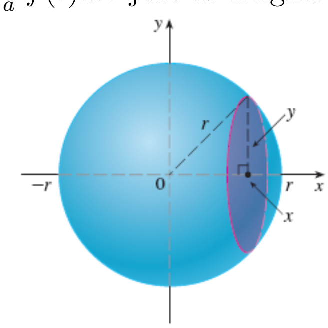
Section 6.2: Volumes

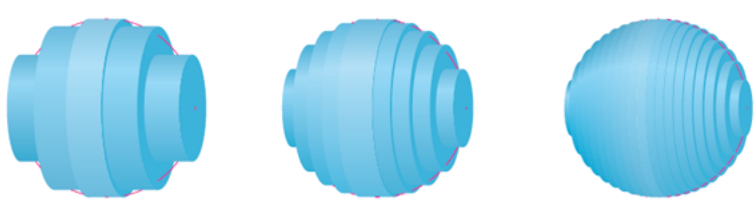
1. In this section, we use integration to find the volume of regions formed by rotating area under curves around a line or volumes of solids with known cross section. Describe how we find the volume of a solid formed by rotating the area under a curve around the x-axis. What are the cross sections? Where do cylinders come in? Where does integration come in?

Solution: Suppose we are taking the volume of the solid formed by rotating the graph of the function $y = f(x)$. We start with estimating the volume, just like we did for area. We split the region up into rectangles. Consider the rectangle whose height is determined by the function value at $x = a$. When we rotate this rectangle, the cross section of the solid is a circle with radius $f(a)$. Thus the volume when this rectangle is rotated around the x-axis is $\pi(f(a))^2 * \Delta x$. We add up all these volumes to get an estimate of the overall volume. We get better and better estimates by letting the number of rectangles go to infinity in which case our sum becomes an integral so we get $V = \int_a^b \pi(f(x))^2 dx$.

Another way to think about this is to integrate cross-sectional areas. We do this for the same reason we integrate along height to find the area under a curve, $A = \int_a^b f(t)dt$: just as heights are infinitesimal areas, areas are infinitesimal volumes.



We can also see cylinders when going back to the Riemann sum definition of the Integral. The left and right rectangle approximations of the volume end up giving cylinders, also known as "discs".



(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

2. How does the process change if we rotate the curve around the y-axis instead? What about if we rotate around a line parallel to the $x-$ or $y-$ axis?

Solution: When rotating around the y-axis, we use the x-coordinate instead of the height (remember that the height is $f(x)$). If f has an inverse then $f^{-1}(t)$ is the x-coordinate at $y = t$. So, if we are integrating along the interval $[a, b]$, the volume is

$$\int_a^b \pi(f^{-1}(y))^2 dy.$$

If f does not have an inverse on $[a, b]$ then we can first partition the interval into $[a = a_0, a_1], [a_1, a_2], ..., [a_{n-1}, a_n = b]$ so that f has an inverse on each part of the partition $[a_{i-1}, a_i]$*. Then, the volume contributed by the ith piece of the partition is

$$\int_{a_{i-1}}^{a_i} \pi(f^{-1}(y))^2 dy.$$

The total volume is then

$$\sum_{i=1}^{n} \left[\int_{a_{i-1}}^{a_i} \pi(f^{-1}(y))^2 dy \right].$$

*note that we need to do more work if $a < 0$ and $b > 0$ to account for overlap in the solid!

If we rotate around a line parallel to the $x-$ or $y-$ axis we merely adjust by the amount the line is rotated: For a function with an inverse on the interval $[a, b]$, the volume of the solid rotated around the line $x = c$ is

$$\int_a^b \pi(f(t) - c)^2 dt.$$

The volume of the solid rotated around the line $y = c$ is

$$\int_a^b \pi(f^{-1}(y) - c)^2 dy.$$

It is very helpful to draw a picture and really think about the length of the radius of the circle you get when rotating.

3. How does the process change if the region we are rotating does not touch the line we are rotating around?

Solution: If the region does not touch the line we are rotating around, we must cut out the portion of the height not being included. On an interval $[a, b]$ where the top of our region is $f(x)$ and the top of our region is $f(x)$, we subtract the volume of the solid formed by rotating $f(x)$ around the $x-axis$ by the volume of the solid formed by rotating $g(x)$ around the $x-axis$. In other words,

$$V = \int_a^b \pi(f(x))^2 dx - \int_a^b \pi(g(x))^2 dx = .$$

Using the fact that you can add and subtract through integrals, we get

$$V = \int_a^b \pi((f(x))^2 - (g(x))^2) dx.$$

We make similar considerations for when functions are rotated around lines parallel to the $y-$ or $x-axis$.
Note that if we are given functions that cross each other on the interval we must construct a suitable partition and treat the larger one appropriately.
Note that if we are rotating around a line parallel to the $y-$axis we also have to adjust our partition so both f and g have inverses on each interval of the parition.

4. How do you find the volume of a solid if you know shape its cross sections are? How does the process change if the cross sections are perpendicular to the x-axis vs the y-axis?

Solution: When we rotated to get the solid, we knew that the cross sections were circles. Then we integrated along these areas to get the volume. Similarly, if we have a formula for area, $A = A(x)$, you can find the volume

$$V = \int_a^b A(x) dx.$$

If the cross sections are perpendicular to the y-axis, we similarly get a formula $A = A(y)$ and integrate

$$V = \int_a^b A(y) dy.$$

Extra Practice in Book: 6.2: 3, 5, 7, 11, 13, 39, 50, 55, 61