Section 3.4: The Chain Rule

(1) In this section, we learn about the chain rule, which allows us to take the derivative of compositions of functions. Give several examples of functions which are compositions of other functions. Also give examples of functions with are not compositions, but are sums, products and quotients. Be sure to indicate which is which.

> There are many possible examples. Here are some functions which are compositions: $\log(x^3+2x+3)$, $\sqrt[3]{\sin(x)}$, $\tan(e^x)$, $\cos\left(\ln\left(\sqrt{x^2-1}\right)\right)$.
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> Functions that are sums: $\log(x)+e^x$, $\sin(x)+\cos(x)$, $3x^2+2$.
>
> Products: $\sin(x)\cdot\cos(x)$, $\log(x)\cdot e^x$.
>
> Quotients:
> $$\frac{\sin(x)}{\cos(x)}, \quad \frac{3x^2+9}{\ln(x)}.$$
>
> Keep in mind that these aren't strict categories: a single function can sometimes be viewed simultaneously as a product, a quotient, and a composition of different functions. For example, consider $f(x)=x^6$. If we let $g(x)=x^2$ and $h(x)=x^3$, then $(g\circ h)(x)=f(x)$, so f is a composition. Additionally, $f(x)=x^3\cdot x^3$, so it is a product. But $f(x)=x^9/x^3$, so it is also a quotient.
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> Because of this, there are often several ways to approach a complicated function, some of which will be easier to use than others.

(2) One of the hardest parts of using the chain rule is recognizing the elementary functions that are being used. Rewrite each function as a composition of basic functions:

(a) $\sin^3(x)$

> $f(x)=x^3$, $g(x)=\sin(x)$, $(f\circ g)(x)=\sin^3(x)$.

(b) $4^{\sin(x)}$

> $f(x)=4^x$, $g(x)=\sin(x)$, $(f\circ g)(x)=4^{\sin(x)}$

(c) $\dfrac{1}{e^{3x}+1}$, two different ways

First way: $f(x) = 1/x, g(x) = e^{3x} + 1$,

$$(f \circ g)(x) = \frac{1}{e^{3x}+1}.$$

Second way: $f(x) = 1/(e^x + 1)$, $g(x) = 3x$,

$$(f \circ g)(x) = \frac{1}{e^{3x}+1}.$$

(3) Write out the chain rule in symbols and in words.

In symbols:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

If $y = f(u)$ and $u = g(x)$, we can also write

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

In words: Derivative of the outer/first function evaluated at the inner/second function, times the derivative of the inner/second function.

(4) Explain how to use the chain rule to exponentiate $y = b^x$.

Recall that $b^x = e^{x \ln b}$. If we let $f(x) = e^x$ and $g(x) = x \ln b$, then we have $(f \circ g)(x) = e^{x \ln b} = b^x$. So

$$\frac{d}{dx}b^x = f'(g(x)) \cdot g'(x) = e^{x \ln x} \cdot \ln x = b^x \ln x.$$

(5) Let $h(x) = f(g(x))$, what values of f' and g' do you need to know to find $h'(x)$? Give an example.

Since $h'(x) = f'(g(x)) \cdot g'(x)$, we need to know the values of $g(x)$, $g'(x)$, and $f'(g(x))$. For example, let $f(x) = x^2$ and $g(x) = \sin(x)$, and suppose we want to evaluate $h'(\pi/4)$. Then we need to find the values of $g(\pi/4)$, $g'(\pi/4)$, and $f(g(\pi/4))$.

$$\begin{aligned} & g(\pi/4) = \sin(\pi/4) = 1/\sqrt{2} \\ g'(x) = \cos(x) : \quad & g'(\pi/4) = \cos(\pi/4) = 1/\sqrt{2} \\ f'(x) = 2x : \quad & f'(g(\pi/4)) = f'(1/\sqrt{2}) = 2/\sqrt{2} \end{aligned}$$

So $h(\pi/4) = f'(g(\pi/4)) \cdot g'(\pi/4) = \frac{2}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{2}{2} = 1$.

Extra Practice in Book: 3.4: Derivative Rules (1-50) until comfortable with all rules. 51, 55, 61, 63, 65, 67, 71, 72, 77