Section 3.3: Derivatives of Trigonometric Functions

(1) In this section, we learn the derivatives of the 6 trig functions. Write out each of these functions and their derivatives.

$$\frac{d}{dx}(\sin(x)) = \cos(x) \qquad x \in \mathbb{R}$$
$$\frac{d}{dx}(\cos(x)) = -\sin(x) \qquad x \in \mathbb{R}$$
$$\frac{d}{dx}(\tan(x)) = \sec^2(x) \qquad x \neq k\pi + \tfrac{\pi}{2}, k \in \mathbb{Z}$$
$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x) \qquad x \neq k\pi + \tfrac{\pi}{2}, k \in \mathbb{Z}$$
$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x) \qquad x \neq k\pi, k \in \mathbb{Z}$$
$$\frac{d}{dx}(\cot(x)) = -\csc^2(x) \qquad x \neq k\pi, k \in \mathbb{Z}$$

(2) Using the derivatives of $\sin(x)$ and $\cos(x)$ and the quotient and chain rules, you can prove the derivative rules for the other four trig functions. Derive the derivative formula for $\sec(x)$ using this method.

$$\begin{aligned}
\frac{d}{dx}(\sec(x)) &= \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) && \text{(By Definition of } \sec(x)\text{)} \\
&= \frac{\frac{d}{dx}(1) \times \cos(x) - \frac{d}{dx}(\cos(x)) \times 1}{\cos^2(x)} && \text{(By Quotient Rule)} \\
&= \frac{\sin(x)}{\cos^2(x)} \\
&= \sec(x)\tan(x) \qquad x \neq k\pi + \frac{\pi}{2}, k \in \mathbb{Z}
\end{aligned}$$

(3) Since $\pm\sin(x)$ and $\pm\cos(x)$ are each others derivatives, if we start taking higher order derivatives, we will notice a repeating pattern. Find a formula for the nth derivative of $\sin(x)$. (You will probably want to use a piecewise function depending on what the remainder is when you divide n by 4).

$$\frac{d}{dx}(\sin^{(n)}(x)) = \cos(x) \qquad n = 4k+1, k \in \mathbb{Z}$$
$$\frac{d}{dx}(\sin^{(n)}(x)) = -\sin(x) \qquad n = 4k+2, k \in \mathbb{Z}$$
$$\frac{d}{dx}(\sin^{(n)}(x)) = -\cos(x) \qquad n = 4k+3, k \in \mathbb{Z}$$
$$\frac{d}{dx}(\sin^{(n)}(x)) = \sin(x) \qquad n = 4k, k \in \mathbb{Z}$$

(4) In this section, we learn that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ and $\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$. Using these limits, we can solve other limits involving trig functions. Explain how you would find the following limit, where m and n are real numbers.

$$\lim_{x\to 0} \frac{\sin(mx)}{nx}$$

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(mx)}{nx} &= \lim_{x\to 0} \frac{\frac{\sin(mx)}{mx}}{\frac{nx}{mx}} && \text{(Divided } mx \text{ on both nominator and denominator)} \\
&= \frac{\lim_{x\to 0} \frac{\sin(mx)}{mx}}{\lim_{x\to 0} \frac{nx}{mx}} && \text{(By Limit Quotient Law)} \\
&= \frac{1}{\frac{n}{m}} \\
&= \frac{m}{n}
\end{aligned}$$

Extra Practice in Book: 3.3: Derivative Rules (1-16) until comfortable with all rules. 19, 21, 29, 31, 33, 39, 51