Logarithms

Since perhaps it's been a while, calculate a few logarithms just to warm up.

1. Calculate the following.
   (a) $\log_3(27) =$

   (b) $\log_9(27) =$

   (c) $\log_3(\frac{1}{9}) =$

   (d) $\ln(e^3) =$

   (e) $\log(-100) =$

   (f) $\ln(0) =$

Just as there are properties of exponents (like $x^a x^b = x^{a+b}$) there are properties of logarithms – in fact, this should be expected since exponentials and logarithms are so closely related. We'll see how we can derive the properties of logarithms now.

2. Let's pick a base just to simplify things – how about base 2. Let's say x is some number, and let's say $X = \log_2(x)$. Let's say y is some other number, and $Y = \log_2(y)$. Finally, let's say $Z = \log_2(xy)$.
   (a) If $Z = \log_2(xy)$, write that as an exponential equation.

   (b) If $X = \log_2(x)$, write that as an exponential equation.

   (c) If $Y = \log_2(y)$, write that as an exponential equation.

   (d) Take your answer to part (a), and substitute in your answers to parts (b) and (c). That is,

   $$2^Z = \underset{\text{(part (a) result)}}{\qquad\qquad} = \underset{\text{(parts (b) and (c) results)}}{\qquad\qquad}$$

   (e) Using rules of exponents, rewrite your result from part (d).

   $$2^Z = 2$$

   (f) From part (e), what can you say about the relationship between $Z, X,$ and Y?

   (g) Substitute back in the definitions of $Z, X,$ and Y. What results is a rule of logarithms.

3. Summarize your result in a general form (since the base being 2 was irrelevant in the previous question).

We won't prove the other rules explicitly here, but we'll talk about why they make sense.

4. The rules of exponents state that $a^{x+y} = a^x a^y$.
   (a) If f is the function $f(x) = a^x$, that means that $f(x + y) =$________.
   (b) In other words, with exponential functions, if you add inputs, that's the same as ____________ outputs.
   (c) Since logarithms and exponential functions are inverses, that's why it makes sense that with logarithmic functions, if you ____________ inputs, that's the same as ____________ outputs.
   (d) The rules of exponents state that $a^{x-y} = \frac{a^x}{a^y}$. In other words, with exponential functions, if you subtract inputs, you ____________ outputs.
   (e) That means that with logarithms, if you ____________ inputs, you ____________ outputs.
   (f) Try to write a rule of logarithms that was just described in the previous question.

5. In this question, we'll develop our next rule.
   (a) What is $\log_a(x \cdot x)$? You can use your result from Question 3 to rewrite this.
   (b) What is $\log_a(x \cdot x \cdot x)$?
   (c) What is $\log_a(x^4)$?
   (d) What do you think $\log_a(x^{38})$ should be?
   (e) Write a rule to summarize.

6. Summarize the rules you've found so far, in Questions 3, 4.f, and 5.e.

    1.

    2.

    3.

Now we'll practice using these a bit. For example, we could rewrite the expression $\log_3(x) - \log_3(y) + 2\log_3(z)$ as follows:

$$\log_3(x) - \log_3(y) + 2\log_3(z) = \log_3\left(\frac{x}{y}\right) + \log_3(z^2) = \log_3\left(\frac{xz^2}{y}\right)$$

7. You try it. Use the rules of logarithms to write the following expressions as logarithms of one quantity with coefficient 1.

    (a) $\frac{1}{2}\ln x + \ln 5$

    (b) $\log_2 x + 4\log_2(x+1) - \frac{1}{3}\log_2(x-1)$

    (c) $5\ln x + 2\ln 3 - 3\ln\left(\frac{1}{y}\right)$

8. What about the "other way?" Use the rules of logarithms to expand the following expressions so that there are no logarithms of products, quotients, or powers.

    (a) $\ln \sqrt[3]{x^3 y}$

    (b) $\log_{10} \frac{10}{4x^2}$

    (c) $\ln\left(\frac{x\sqrt{y}}{(1+x)^3}\right)$

9. Now use your critical thinking skills and the rules we've learned. Suppose $\ln x = 2$, $\ln y = 3$ and $\ln z = 6$. Evaluate the following.

   (a) $\ln(xyz)$

   (b) $\ln(x^2 y)$

   (c) $\ln\left(\frac{x^3}{\sqrt{z}}\right)$

Our last set of properties involves changing the base of a logarithm or exponential function.

10. Can you simplify $3^{x \log_3(5)}$? (First, try to change the expression in the exponent.)

11. What about $e^{x \ln(7)}$?

12. Let's say I have 4^5, and I want to write that as an expression with base 3 instead? That is, I want to write
$$4^5 = 3^{\text{something}}.$$
Let's figure out how to do this.

   (a) Fill in the blank: $4^5 = 3^{\log_3(\text{—})}$.

   (b) Use rules of logarithms to rewrite your exponent.

13. In general, if you have a^x, and you want to write that as $b^{\text{something}}$, you can do this. Write down the rule below.

14. Now let's see what the rule would be for logarithms, just by analogy. For exponentials, if you have a^x, and you want to write this with base b, you _______________ the input by a factor of ____________.

15. For logarithms, by analogy, if you have $\log_a(x)$, an you want to write this using logarithms with base b, you should _______________ the output by a factor of ____________.

16. The *change of base formula* for logarithms is:

Let's use these a bit.

17. Write $\log_3(5)$ as a logarithm with base 2.

18. Write $\ln(x)$ as a logarithm with base 10.

19. Simplify the expression $\log_3(5) + \log_9(5)$.

20. Write 5^x as an exponential with base e.

21. Write 2^7 as an exponential with base 10.

22. Write x^x as an exponential with base e.

23. Summarize the rules of logarithms so that you can remember them, making any notes to help you do so!