Logarithms

A *logarithmic function* is a function that is the inverse of an exponential function. Since exponential functions have different bases, so do logarithms. The logarithm, with base a, of x, is written $\log_a(x)$. Sometimes we leave off the parentheses because we're lazy, but we shouldn't be lazy – parentheses are good and important and if you leave them off in the wrong place it's wrong.

1. So logarithms are the inverses of exponential functions. Let $f(x) = 2^x$. Then $f^{-1}(x) = \log_2(x)$. Of course, that doesn't tell you anything about how to compute values of $\log_2(x)$.
   (a) Since $f(1) = 2$, $f^{-1}(2) = 1$. In other words, $\log_2(2) =$
   (b) Since $f(2) = 4$, $\log_2(4) =$
   (c) Since $f(3) = 8$, $\log_2(8) =$
   (d) $\log_2(16) =$
   (e) $\log_2(32) =$
   (f) $\log_2(1) =$
   (g) $\log_2(\sqrt{2}) =$
   (h) $\log_2(\frac{1}{2}) =$
   (i) $\log_2(\frac{1}{4}) =$
   (j) $\log_2(0) =$
   (k) $\log_2(-2) =$
   (l) $\log_2(10) =$ (Here, you won't be able to get an exact answer without a calculator. Try to approximate if you can.)

Often, it's easiest to deal with logarithms by translating between "logarithmic form" and "exponential form" of an equation.

2. If $\log_a(x) = y$, then $?^? = ?$. Fill in the three question marks with a, x, and y in the appropriate spots.

3. That last question is important and you should remember it! This isn't a new question. It's just telling you to go back and look at the last one again.

4. One more time...try to find a way to remember this relationship so you don't have to derive it all the time.

5. Okay, moving on. Calculate some more logarithms. Some of these require thoughtful guesswork.
   (a) $\log_3(9) =$
   (b) $\log_3(-3) =$
   (c) $\log_5(125) =$
   (d) $\log_{10}(.1) =$
   (e) $\log_4(2) =$
   (f) $\log_4(8) =$
   (g) $\log_{\frac{1}{9}}(9) =$

A few bases come up so often, they're considered special and have their own notation. Logarithms with base 10 are typically denoted just $\log(x)$ – in other words, if you don't see a base, you can assume the base is 10. Another common base is the *natural base*, e. This is a number, $e \approx 2.71828\ldots$, that comes up in mathematics naturally (one place, as you'll see later, is when dealing with compound interest). The logarithm with base e is denoted $\ln(x)$ and is called the *natural logarithm*.

6. Calculate a few more:
   (a) $\log(10) =$
   (b) $\log(100000000) =$
   (c) $\ln(e) =$
   (d) $\ln(e^7) =$
   (e) $\ln(\sqrt{e}) =$

7. In these last few problems, you're already starting to discover some properties of logarithms. Let's formalize one of them.
   (a) Can you simplify $\log_a(a^x)$?

   (b) There's a reason this makes sense. What's the relationship between the two functions $f(x) = a^x$ and $g(x) = \log_a(x)$?

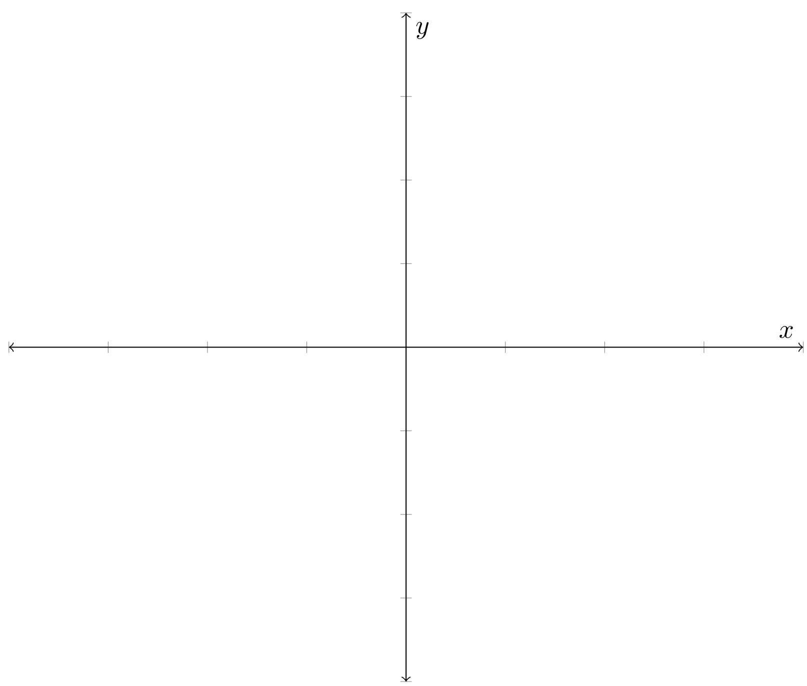
   (c) Think back to inverse functions: if you have a function f and its inverse f^{-1}, what is $(f^{-1} \circ f)(x)$?

   (d) In the case where $f = a^x$, what is $(f^{-1} \circ f)(x)$? Write it out.

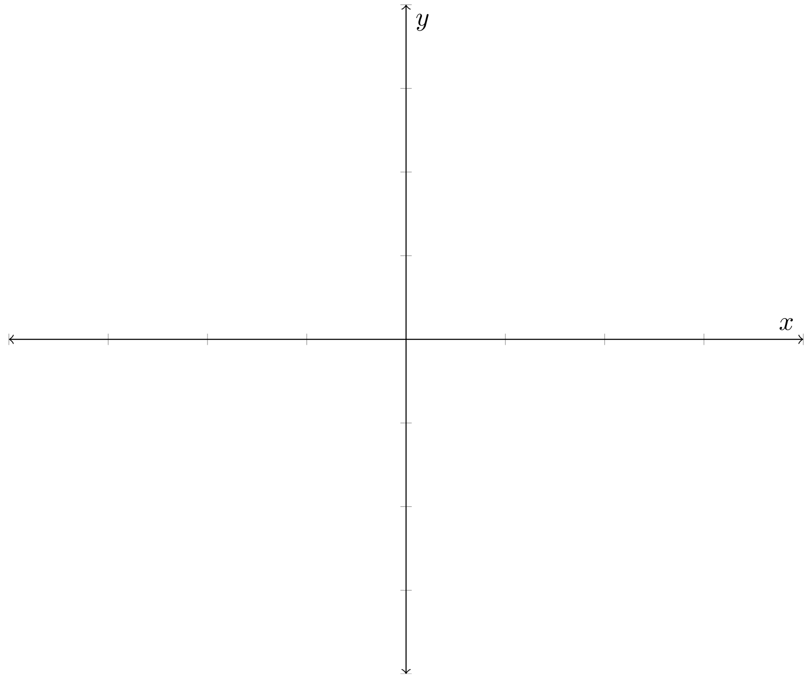
   (e) In the case where $f = a^x$, what is $(f \circ f^{-1})(x)$? Can you derive another property of logarithms?

Let's finish up with the graphs of logarithmic functions. They're easy to derive if you remember the graphs of exponential functions.

8. On the axes below, draw the graph of the function $f(x) = 2^x$, the line $y = x$, and the graph of the function $f^{-1}(x) = \log_2(x)$.

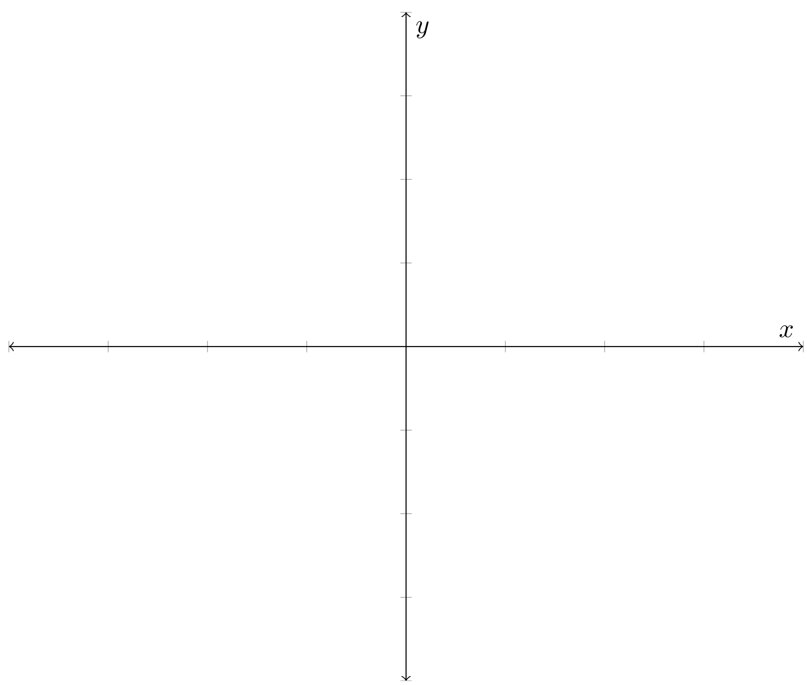


9. On the axes below, draw the graph of the function $g(x) = \log_3(x)$. Write labels with the coordinates of 3 points on this graph.



10. Every function $f(x) = \log_a(x)$ passes through the point $(\underline{\quad}, 0)$.

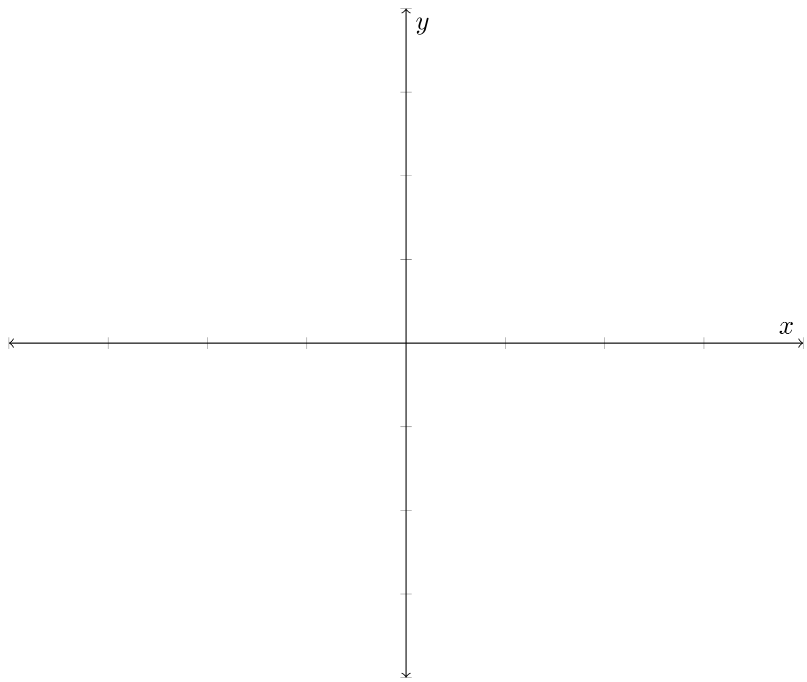
11. So what's the difference between the graphs of logarithms with different bases? On the axes below, sketch the graphs of $\log_2(x), \log_3(x)$, and $\ln(x)$.



12. What's the domain of the function $f(x) = \ln(x)$? What's the range?

Finally, let's just play around a bit with graph transformations.

13. Sketch a graph of $h(x) = \ln(x-2)$. What is the domain of this function? What is its range?



14. Sketch a graph of $g(x) = -\log(x+1) - 2$. What is the domain of this function? What is its range?