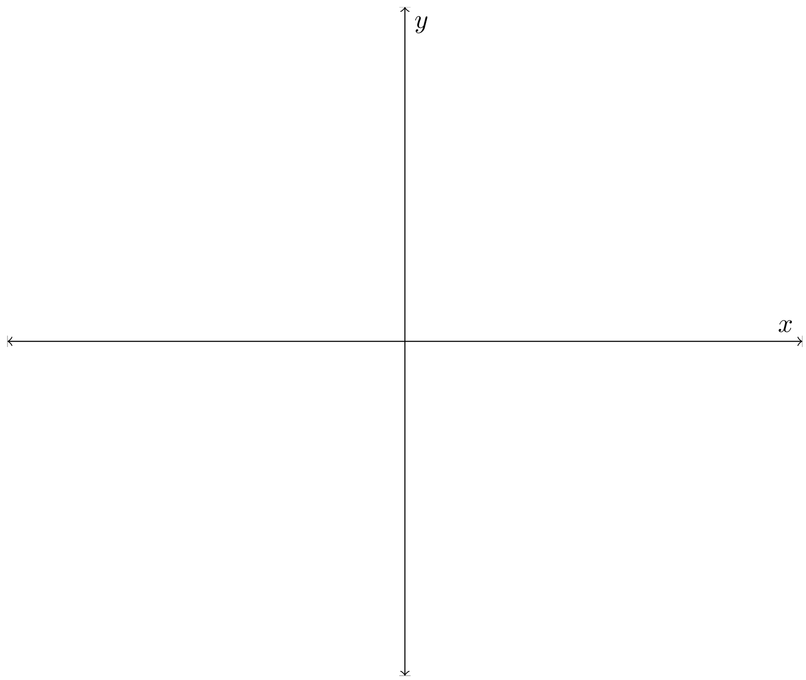
Inverse Trigonometric Functions

1. Trig functions don't really have inverses, because they're not ________.

However, we can find partial inverses by restricting the domain.

2. Let's say we want to find the inverse sine function.
   (a) Draw a graph of $\sin(x)$.



   (b) Pick an interval $[a, b]$ such that $\sin(x)$ is 1-1 on that interval. There are options – try to pick the most reasonable option.

   (c) What is the domain and range of $\sin(x)$ on this interval?

   (d) What should the domain and range of the inverse of $\sin(x)$ be? (We write $\sin^{-1}(x)$ or $\arcsin(x)$ for the inverse of $\sin(x)$. Those two notations are equivalent.)

(e) We're going to try to graph $\sin^{-1}(x)$. First, let's figure out a few points. Fill in the blanks:

(a) $\sin(0) = \quad$, so $\sin^{-1}(\quad) = \quad$.

(b) $\sin(\frac{\pi}{6}) = \quad$, so $\sin^{-1}(\quad) = \quad$.

(c) $\sin(\frac{\pi}{4}) = \quad$, so $\sin^{-1}(\quad) = \quad$.

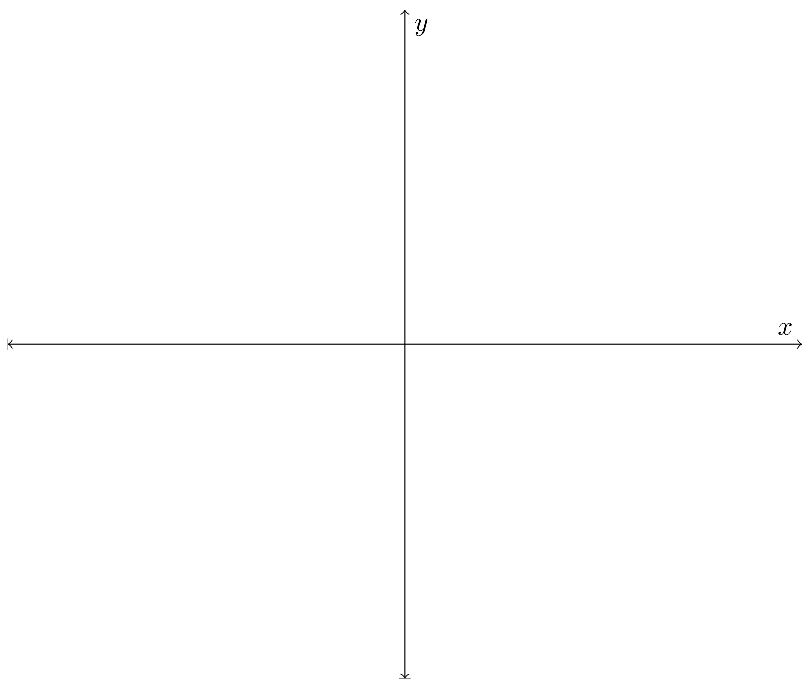
(d) $\sin(\frac{\pi}{2}) = \quad$, so $\sin^{-1}(\quad) = \quad$.

(e) $\sin(-\frac{\pi}{6}) = \quad$, so $\sin^{-1}(\quad) = \quad$.

(f) $\sin(-\frac{\pi}{4}) = \quad$, so $\sin^{-1}(\quad) = \quad$.

(g) $\sin(-\frac{\pi}{2}) = \quad$, so $\sin^{-1}(\quad) = \quad$.

(h) Sketch a graph of $\arcsin(x)$.



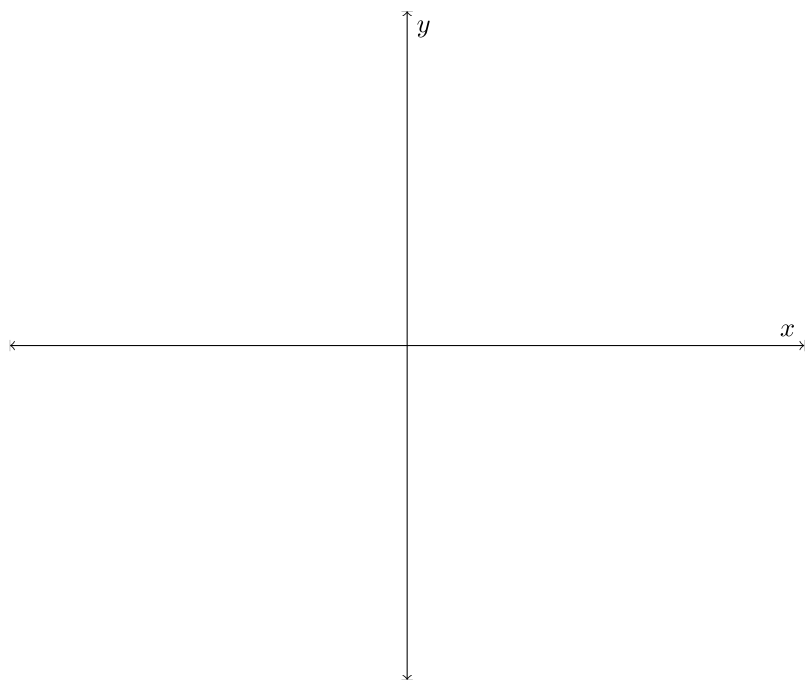
3. Now let's do the same for $\cos(x)$ and $\arccos(x)$.

(a) What would a good restricted domain be for $\cos(x)$ to make it 1-1?

(b) What should the domain and range of $\cos^{-1}(x)$ be?

(c) Find some points on the graph of $\cos^{-1}(x)$.

(d) Sketch a graph of $\cos^{-1}(x)$.



Before doing the same thing for $\tan^{-1}(x)$, let's talk a little more about computing inverse trig functions. Remember – for trig functions, you plug in an angle and get out a ratio/number, which means for inverse trig functions, you plug in a ratio/number and get out an angle. What makes them a little bit more difficult is the restrictions we have in place, because trig functions are not 1-1.

4. What is $\arccos(1)$?

5. What is $\arccos(-1)$?

6. What is $\arcsin(\frac{1}{2})$?

7. What is $\arcsin(-\frac{\sqrt{2}}{2})$?

8. What is $\arcsin(-1)$?

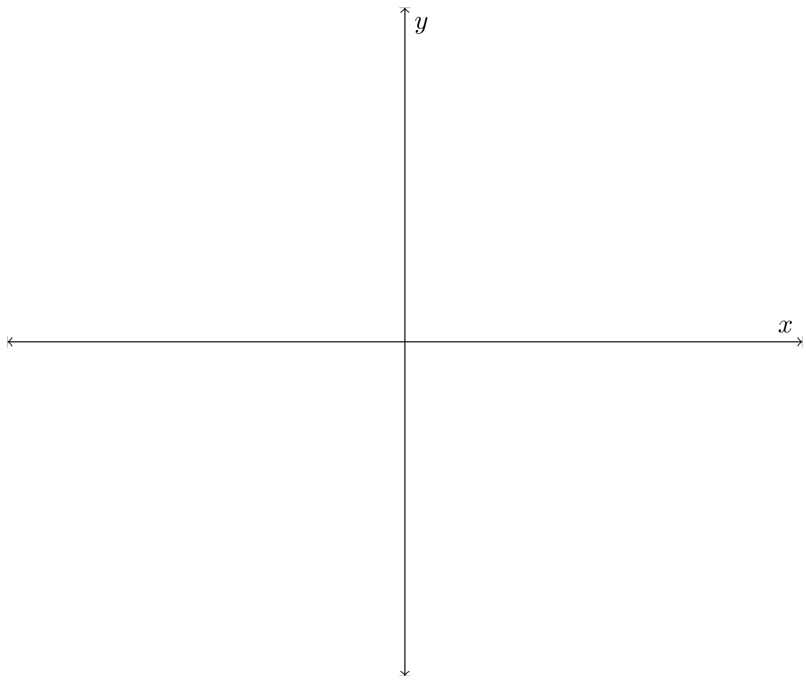
9. What is $\arccos(-\frac{1}{2})$?

10. What is $\arctan(1)$?

11. Let's get back to graphing, and find the graph of $\arctan(x)$.

    (a) Find a domain on which $\tan(x)$ is 1-1.

    (b) Sketch a graph of $\arctan(x)$.



Remember how functions and their inverses should cancel each other out? That is, $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$. Well, when we deal with partial inverses and restricted domains, things can get a little screwy. Let's just see what might happen.

12. Start with a non-trig example. Hopefully you remember that $\sqrt{x}$ is the partial inverse of $f(x) = x^2$.

    (a) We can compute $(f^{-1} \circ f)(x)$, which in this case is $\sqrt{x^2}$. If $x = 4$, what is $(f^{-1} \circ f)(x)$?

    (b) What is $(f \circ f^{-1})(x)$, if $x = 4$?

    (c) Hopefully the first two parts worked out nicely. However, what if $x = -4$? Then what is $(f^{-1} \circ f)(x)$?

    (d) What is $(f \circ f^{-1})(x)$, if $x = -4$?

These weird things happen because $\sqrt{x}$ is only a partial inverse of x^2. Similar weird things can happen when you compose trig functions and their inverses. Sometimes it works out nicely, but sometimes not...

13. What is $\sin(\arcsin(1))$?

14. What is $\sin(\arcsin(-\frac{1}{2}))$?

15. What is $\sin^{-1}(\sin(0))$?

16. What is $\arcsin(\sin(\frac{2\pi}{3}))$?

17. What is $\sin^{-1}(\sin(\frac{\pi}{3}))$?

18. What is $\arcsin(\sin(\frac{11\pi}{6}))$?

19. What is $\cos(\arccos(-\frac{1}{2}))$?

20. What is $\cos^{-1}(\cos(3\pi))$?

21. What is $\tan(\arctan(\sqrt{3}))$?

22. What is $\tan^{-1}(\tan(\pi))$?

23. What is $\tan^{-1}(\tan(-\frac{\pi}{3}))$?

One thing that comes up in calculus is simplifying expressions that involves trig and inverse trig functions. For example, you might want to simplify the expression $\sin(\arctan(\frac{x}{3}))$. This isn't too hard to do, with the aid of a triangle.

24. Consider $\arctan(\frac{x}{3})$. This quantity is an angle, because the arctan function spits out angles. Call this angle θ. Draw a right triangle, with θ as one of the angles.

25. If $\theta = \arctan(\frac{x}{3})$, then $\tan(\quad) = \quad$.

26. Label the sides of the triangle you drew with expressions involving x and 3. The Pythagorean Theorem may come in handy.

27. What is $\sin(\arctan(\frac{x}{3}))$?

28. What is $\tan(\arccos(\frac{1}{x}))$?

29. What is $\csc(\sin^{-1}(x))$? (This is an easy one...why?)

30. What is $\sec(\sin^{-1}(x))$?