Exponential and Logarithmic Equations

Solving exponential and logarithmic equations is a lot like solving trigonometric equations. Imagine you have an equation like $f(x)^2 - f(x) - 2 = 0$, where $f(x)$ might be $\sin(x)$, $\tan(x)$, e^x, $\log_2(x)$, or any other function. In any of these cases, your goal is to isolate $f(x)$:

$$(f(x) - 2)(f(x) + 1) = 0 \implies f(x) = 2 \text{ or } f(x) = -1.$$

Then, if f were a trig function, you'd go to the unit circle to solve. In the case of exponential or logarithmic functions, you can "logarithmify" or "exponentiate" both sides of the equation to solve.

1. Let's work through one together. Let's say we want to solve the equation $2\log_9(x) - 2 = 1$.
   (a) Simplify this equation until the logarithm is isolated. In other words, simplify it until it is of the form "$\log_9(x) =$ something."

   (b) Just as you can add the same thing to both sides of an equation, or multiply both sides of an equation by the same quantity, you can also make both sides of an equation an exponent with the same base. For example, if $3x = 5$, then $2^{3x} = 2^5$. (This operation is not always useful, of course.) Apply this operation to the simplified equation from part (a), with base 9.

   (c) Simplify the left hand side using rules of exponents and logarithms.

   (d) What is x?

2. Now let's solve an exponential equation. Say $2e^x - 2 = 1$.
   (a) Simplify to isolate the exponential term.

   (b) Another operation we can use to solve equations is to take the logarithm (with the same base) of both sides. Take the natural logarithm of both sides of your simplified equation.

   (c) Simplify both sides of the equation, and solve for x.

Now solve some more. The initial algebra may be a little more complicated (remember, factoring can be useful!) but the basic idea is the same for these equations.

3. $6^{2x+1} = 5$

4. $\ln x = -1$

5. $\log_3(x-1) = 2$

6. $3 \cdot 2^{x+1} = 9$

7. $(\log(x))^2 - 3\log(x) = 0.$

8. $3^{2x} - 3^x - 2 = 0$ (Here's a hint: $3^{2x} = (3^x)^2$.)

9. This one's a little different, but you can figure it out with the knowledge you have learned!

$$\log_3(\ln x) = 2$$

Some equations can get a little more complicated. The rules of logarithms could come in handy in these equations below. It's especially important to check your answers in these types of equations, as you can often arrive at "false" solutions, even if you don't make any mistakes.

10. Solve $\log_2 x + \log_2(x-3) - 4 = -2$.

11. Solve $\log_3 x = 3 - \log_3(x+6)$.

12. Solve $\log_2(x+1) - \log_2(x-1) - 3 = 0$.

We can even deal with different bases, as long as we remember rules of exponents and logarithms. Let's do an example. Say we want to solve $2^x = 3^{x+1}$. We could "logarithmify" this equation with either base 2 or base 3; our answers will look different, but will be equivalent. If we use base 2, we would get:

$$\begin{aligned} 2^x &= 3^{x+1} \\ \log_2(2^x) &= \log_2(3^{x+1}) \\ x &= (x+1)\log_2(3) \\ x &= x\log_2(3) + \log_2(3) \\ x - x\log_2(3) &= \log_2(3) \\ x(1 - \log_2(3) &= \log_2(3) \\ x &= \frac{\log_2(3)}{1 - \log_2(3)} \end{aligned}$$

Or, we could use base 3:

$$\begin{aligned} 2^x &= 3^{x+1} \\ \log_3(2^x) &= \log_3(3^{x+1}) \\ x\log_3(2) &= x + 1 \\ x\log_3(2) - x &= 1 \\ x(\log_3(2) - 1) &= 1 \\ x &= \frac{1}{\log_3(2) - 1} \end{aligned}$$

Although the answers look different, they are equivalent (a good exercise would be to try to write one so that it looks like the other, using properties of logs and the change of base formula).

13. Solve $5^x = 2^{-x}$.

14. Solve $2 \cdot 2^x = 5$.

15. Solve $300(1.05)^t = 600$.

16. Solve $3 \cdot 4^x = 3^{x-2}$.