Absolute Value

Most of you have seen absolute value before. You might know it as the two little lines that make everything positive. That's a good start, but there's more to the story of absolute value, which we'll start to explore in this worksheet. Start by simplifying a few expressions that involve absolute value:

1. $|3| =$
2. $|-5| =$
3. $|178| =$
4. $|0| =$
5. $|-95| =$

Hopefully that was easy. The hard part comes when we have something unknown inside the absolute value signs, like a variable, x. From the work above, can you come up with a rule that describes $|x|$ in words?

6. The absolute value of x is ________ if x is positive, and is ________ if x is negative.
7. Let's modify this just a bit, since we've left out 0, which is neither positive nor negative. We can write it this way: The absolute value of x is ________ if x is nonnegative, and is ________ if x is negative.

So it seems part of the key to absolute value is the "if"s. That is, if we have to deal with an absolute value that has an unknown, maybe we can divide it up into cases – "well, *if* this is true, then what happens, and *if* it's not true, then what? Let's try an example.

8. Follow the reasoning below to solve the equation. That is, find all possible values for x that make the equation true.

$$|x - 4| = 12$$

   (a) The thing inside the absolute value is "$x - 4$." If $x - 4$ is nonnegative, what is $|x - 4|$? We're not expecting a number here (we still don't know what x is), but you can rewrite the expression $|x - 4|$ without any absolute value signs if you assume $x - 4$ is nonnegative. (Don't overthink it, this answer is pretty simple!)

   (b) If $x-4$ is negative, then what is $|x-4|$? Again, rewrite the expression without any absolute value signs.

   (c) We can use this to divide this equation into two cases:

      (i) If $x - 4$ is nonnegative, then we can rewrite the equation as:

      (ii) If $x - 4$ is negative, then we can rewrite the equation as:

   (d) Solve each of those equations to find two possible solutions for x. Check your answers.

9. Before we go on, let's just hit upon a common mistake (if you made this mistake, hopefully you noticed it when you checked your answers in the previous problem!). What is $-(x-5)$?
   (a) $-x-5$
   (b) $-x+5$
   (If you're not sure, just plug in a value for x to check!) Watch out for this common mistake!

Now solve the equations below, using this technique of "dividing up into cases." Remember to check your answers!

10. Solve the equation.
$$|4x+1| = 3$$

11. Solve the equation.
$$|4x+1| = 3x$$

12. Solve the equation.
$$|2-2x| = 100$$

13. Solve the equation.
$$|1-x| = x$$

What about some inequalities?

14. Solve the inequality. That is, find all possible values for x that make the inequality true. Remember, divide things up into cases, and write your answer using interval notation.

$$|x - 4| < 12$$

15. Solve the inequality.

$$|x - 2| < .5$$

16. These particular types of inequalities come up fairly often in calculus; that is, inequalities of the form $|x - a| < \delta$. They can be interpreted in a nice way.
    (a) For the inequality above, $|x - 2| < .5$, draw your solution set on the number line below.

    (b) Then you can see that the solution set to this inequality is all values of x such that ______ is within ______ of ______.
    (c) In general, the solution set to $|x - a| < \delta$ is the set of all values of x such that ______ is within ______ of ______.

There are an infinite number of "types" of absolute value equations we might need to solve, but we'll just do one more simple one.

17. Solve the inequality.

$$|2x - 3| > 4$$

This might be a good time to see what the properties of absolute value might be. Absolute value is hard to manipulate algebraically, and your best tool is dividing things into cases. Let's see what we *can* do. Things like the distributive law, factoring, adding absolute values, how do they work? Time to explore. Explain your reasoning!

18. True or False: $|x| + |y| = |x + y|$

19. True or False: $|x||y| = |xy|$

20. True or False: $|-x| = |x|$

21. True or False: $|2x + 2y| = 2|x + y|$

22. True or False: $|-3x - 6| = -3|x + 2|$

23. True or False: $|x^2| = |x|^2$

24. True or False: $|x + y|^2 = |x^2 + y^2|$