Logarithms

A *logarithmic function* is a function that is the inverse of an exponential function. Since exponential functions have different bases, so do logarithms. The logarithm, with base a, of x, is written $\log_a(x)$. Sometimes we leave off the parentheses because we're lazy, but we shouldn't be lazy – parentheses are good and important and if you leave them off in the wrong place it's wrong.

1. So logarithms are the inverses of exponential functions. Let $f(x) = 2^x$. Then $f^{-1}(x) = \log_2(x)$. Of course, that doesn't tell you anything about how to compute values of $\log_2(x)$.
   (a) Since $f(1) = 2$, $f^{-1}(2) = 1$. In other words, $\log_2(2) =$

   (b) Since $f(2) = 4$, $\log_2(4) =$

   (c) Since $f(3) = 8$, $\log_2(8) =$

   (d) $\log_2(16) =$

   (e) $\log_2(32) =$

   (f) $\log_2(1) =$

   (g) $\log_2(\sqrt{2}) =$

   (h) $\log_2(\frac{1}{2}) =$

   (i) $\log_2(\frac{1}{4}) =$

   (j) $\log_2(0) =$

   (k) $\log_2(-2) =$

   (l) $\log_2(10) =$ (Here, you won't be able to get an exact answer without a calculator. Try to approximate if you can.)

Often, it's easiest to deal with logarithms by translating between "logarithmic form" and "exponential form" of an equation.

2. If $\log_a(x) = y$, then $?^? = ?$. Fill in the three question marks with a, x, and y in the appropriate spots.

3. That last question is important and you should remember it! This isn't a new question. It's just telling you to go back and look at the last one again.

4. One more time...try to find a way to remember this relationship so you don't have to derive it all the time.

5. Okay, moving on. Calculate some more logarithms. Some of these require thoughtful guesswork.
   (a) $\log_3(9) =$
   (b) $\log_3(-3) =$
   (c) $\log_5(125) =$
   (d) $\log_{10}(.1) =$
   (e) $\log_4(2) =$
   (f) $\log_4(8) =$
   (g) $\log_{\frac{1}{9}}(9) =$

A few bases come up so often, they're considered special and have their own notation. Logarithms with base 10 are typically denoted just $\log(x)$ – in other words, if you don't see a base, you can assume the base is 10. Another common base is the *natural base*, e. This is a number, $e \approx 2.71828\ldots$, that comes up in mathematics naturally (one place, as you'll see later, is when dealing with compound interest). The logarithm with base e is denoted $\ln(x)$ and is called the *natural logarithm*.

6. Calculate a few more:
   (a) $\log(10) =$
   (b) $\log(100000000) =$
   (c) $\ln(e) =$
   (d) $\ln(e^7) =$
   (e) $\ln(\sqrt{e}) =$

7. In these last few problems, you're already starting to discover some properties of logarithms. Let's formalize one of them.
   (a) Can you simplify $\log_a(a^x)$?
   (b) There's a reason this makes sense. What's the relationship between the two functions $f(x) = a^x$ and $g(x) = \log_a(x)$?
   (c) Think back to inverse functions: if you have a function f and its inverse f^{-1}, what is $(f^{-1} \circ f)(x)$?
   (d) In the case where $f = a^x$, what does that mean? Write it out.

(e) In the case where $f = a^x$, what is $(f \circ f^{-1})(x)$? Can you derive another property of logarithms?

Those properties you just found are the inverse properties of logarithms and exponents. They tell you how the two work together. There are more properties of logarithms. Since there are properties of exponents, and exponentials and logarithms are related, this makes sense. We'll come up with them now.

8. Let's pick a base just to simplify things – how about base 2. Let's say x is some number, and let's say $X = \log_2(x)$. Let's say y is some other number, and $Y = \log_2(y)$. Finally, let's say $Z = \log_2(xy)$.

   (a) If $Z = \log_2(xy)$, write that as an exponential equation.

   (b) If $X = \log_2(x)$, write that as an exponential equation.

   (c) If $Y = \log_2(y)$, write that as an exponential equation.

   (d) Take your answer to part (a), and substitute in your answers to parts (b) and (c). That is,

   $$2^Z = \underset{\text{(part (a) result)}}{\rule{3cm}{0pt}} = \underset{\text{(parts (b) and (c) results)}}{\rule{3cm}{0pt}}$$

   (e) Using rules of exponents, rewrite your result from part (d).

   $$2^Z = 2$$

   (f) From part (e), what can you say about the relationship between Z, X, and Y?

   (g) Substitute back in the definitions of Z, X, and Y. What results is a rule of logarithms.

9. Summarize your result in a general form (since the base being 2 was irrelevant in the previous question).

We won't prove the other rules explicitly here, but we'll talk about why they make sense.

10. The rules of exponents state that $a^{x+y} = a^x a^y$.
    (a) If f is the function $f(x) = a^x$, that means that $f(x+y) =$________.
    (b) In other words, with exponential functions, if you add inputs, that's the same as __________ outputs.
    (c) Since logarithms and exponential functions are inverses, that's why it makes sense that with logarithmic functions, if you __________ inputs, that's the same as __________ outputs.
    (d) The rules of exponents state that $a^{x-y} = \frac{a^x}{a^y}$. In other words, with exponential functions, if you subtract inputs, you __________ outputs.
    (e) That means that with logarithms, if you __________ inputs, you __________ outputs.
    (f) Try to write a rule of logarithms that was just described in the previous question.

11. In this question, we'll develop our next rule.
    (a) What is $\log_a(x \cdot x)$? You can use your result from Question 9 to rewrite this.
    (b) What is $\log_a(x \cdot x \cdot x)$?
    (c) What is $\log_a(x^4)$?
    (d) What do you think $\log_a(x^{38})$ should be?
    (e) Write a rule to summarize.

12. Summarize the rules you've found so far, in Questions 7.d, 7.e, 9, 10.f, and 11.e.

    1.

    2.

    3.

    4.

    5.

Our last set of properties involves changing the base of a logarithm or exponential function.

13. Can you simplify $3^{x \log_3(5)}$?

14. What about $e^{x \ln(7)}$?

15. Let's say I have 4^5, and I want to write that as an expression with base 3 instead? That is, I want to write

$$4^5 = 3^?$$

    What would the ? need to be?

16. In general, if you have a^x, and you want to write that as $b^{\text{something}}$, you can do this. Write down the rule below.

17. Now let's see what the rule would be for logarithms, just by analogy. For exponentials, if you have a^x, and you want to write this with base b, you ________________ the input by a factor of ____________.

18. For logarithms, by analogy, if you have $\log_a(x)$, an you want to write this using logarithms with base b, you should ________________ the output by a factor of ____________.

19. The *change of base formula* for logarithms is: