Algebra

This is a review of some algebra skills you already have, but might need refreshed.

Why parentheses are so important. A little story.

1. Lassie barks at Timmy: "Timmy, you need to compute 12 minus 3 plus 5!" Timmy does this (he's good at math). What does he get?

2. Lassie barks at Timmy: "That's not what I got!" What did Lassie find for the answer?

3. What's the problem?

Use parentheses! Otherwise, you're ambiguous, and if you're ambiguous, it's wrong. (Would you like an ambiguous pharmacist? Surgeon? Hair stylist?)

4. Is there a difference between $(3x)^4$ and $3x^4$?
5. Is there a difference between $(3 + x)^4$ and $3^4 + x^4$?
6. Add parentheses to $3x - 2^2$ in three different ways to make it mean three different things.

$$3\,x - 2\,^2 \qquad\qquad 3\,x - 2\,^2 \qquad\qquad 3\,x - 2\,^2$$

Fractions. The big idea: multiplication is easy, division's alright, adding and subtracting is a pain.

7. When do you need a common denominator?

8. How do you multiply fractions?

9. How do you divide fractions?

10. When do you "cross-multiply" fractions?

11. What is $3 + \frac{7}{5}$ as a single fraction? (Please – no "mixed fractions" like $7\frac{12}{17}$. We will not be using those in this class.)

12. What is $\frac{3}{x} + \frac{5x}{y}$ as a single fraction?

13. What is $\frac{x}{x+1} + \frac{x+1}{x}$ as a single fraction?

14. What is $\frac{12z}{3x^2} - \frac{y+2}{xz}$ as a single fraction?

15. True or false? $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$. Explain your reasoning.

16. True or false? $\dfrac{a}{b+c} = \dfrac{a}{b} + \dfrac{a}{c}$. Explain your reasoning.

17. What is $\frac{7}{\frac{3}{5}}$?

18. Is $\frac{a}{\frac{b}{c}}$ the same as $\frac{\frac{a}{b}}{c}$? Explain your answer.

19. What is $\frac{\frac{3}{\frac{x}{x}}}{6} \cdot \frac{1}{x^3}$?

Distributive Law. Ever hear of the distributive law? Hopefully so. It says that $a(b+c) = ab+ac$, or $(a+b)c = ac+bc$. Basically, it tells you how addition and multiplication work together. You can *distribute* multiplication over addition.

20. What is $x(x+3)$? (In other words, multiply it out – what do you get?)

21. What is $(x+3)^2$? (Hint: if you say x^2+9, you are wrong. WRONG.)

22. Give a justification as to why it's *not* x^2+9.

23. Is there a distributive law with exponents?

24. Why do we FOIL? Why does it work?

25. How would you multiply out $(a+b+c)(d+e)$?

26. What about $(a+b)^3$?

Exponents. Let's just make sure we're all on the same page as far as exponents.

27. How can $a^b a^c$ be simplified, and why is that true?

28. How can $(a^b)^c$ be simplified, and why is that true?

29. How can $\frac{a^b}{a^c}$ be simplified, and why is that true?

30. How can $(ab)^c$ be simplified, and why is that true?

31. What's another way to write a^{-b}, and why does that make sense?

32. Is $\left(a^b\right)^c$ the same as $a^{\left(b^c\right)}$? Why or why not?

33. What's wrong with the following? Included are actual, common mistakes taken from previous exams – find as many as you can.

$$\left(\frac{u^{-1}v^3}{2w^3}\right)^{-3}(vw^2) = \left(\frac{u^3v^{-9}}{2w^{-9}}\right)(vw^2) = \frac{u^3v^{-8}}{2w^{-7}} = \frac{u^3v^{\frac{1}{8}}}{2w^{\frac{1}{7}}}$$

Factoring.

34. What's the relationship between factoring and the distributive law?

35. Factor $x^3 - 5x^2 + 4x$ as much as possible.

36. Factor $2z^2 - 11z + 5$ as much as possible.

37. Factor $3x^2y - \frac{6y}{x}$ as much as possible.

AB = 0. Factoring is useful, in large part, because it helps us solve equations. It helps us solve equations because of one key fact. If $AB = 0$, then either $A = 0$ or $B = 0$. This is sometimes called the "Zero-Factor Theorem," or "Zero-Product Property," or something like that.

38. Is there something special about 0? Is it true that if $AB = 1$, then $A = 1$ or $B = 1$?

39. How does this property come into play when solving $x^2 - 3x - 4 = 0$?

40. Can you use this property to solve $\frac{x^{\frac{3}{2}}}{3} + 4\sqrt{x} = 0$?

41. What about $(y-1)^2 = 2(y-1)$?

Interval Notation. Often, we need to describe sets of real numbers. Like "all real numbers between 3 and 7." So we use shorthand "interval notation."

42. Written out in words, something like $[a, b]$ means "all real numbers x such that $a \leq x \leq b$." That is, $[a, b]$ refers to a whole set of numbers. Is 3 in the set $[-2, 4]$?

43. Something like (a, b) means "all real numbers x such that $a < x < b$." Is 5 in the set $(-5, 5)$?

44. How would you write the set "all real numbers x such that $2 \leq x < 18$?

45. How are you going to remember the difference between square brackets, $[,]$ and parentheses $(,)$?

46. Occasionally, students will think that $(3, 7)$ is the same as $[4, 6]$. Why do they think this and why are they wrong?

47. If we want to just say "all real numbers x such that $x > -2$," we can write that as $(-2, \infty)$. How would you write "all real numbers x less than or equal to 5?"

48. How would you write "all real numbers" using interval notation? (Note: the set of all real numbers is also sometimes written $\mathbb{R}$.)

We can combine intervals in a few ways. One way is by taking the *union* of two intervals, which means you take the set of all real numbers that are in one interval or the other. This is written as $\cup$ (like a U, but without any tail). For example, $(-2, 0) \cup [3, 5]$ is the set of numbers x such that $-2 < x < 0$ *or* $3 \leq x \leq 5$. That is, -1 is in this set, and $-.2$ is in this set, and π is in this set, and 5 is in this set…

49. How would you write the set of numbers x such that x is negative or x is larger than 100?

50. Can you write the set $(-\infty, 3] \cup (2, 7)$ as a single interval?

51. Using this notation, how would you write the set of all real numbers x such that $x \neq 3$?

Another way to combine intervals is through *intersection*. The intersection of two intervals, written with a $\cap$, is the set of all numbers that are in one interval *and* the other. For example, $(-2, 4) \cap [3, 4]$ contains the number 3.5, since 3.5 is in both intervals, but it does not contain the number 4, since 4 is in $[3, 4]$ but is not in $(-2, 4)$.

52. Can you write $(-2, 4) \cap [3, 4]$ as a single interval?

53. What about $(-3, 4] \cap (-1, 5)$?

54. What about $[0, 1] \cap [1, \infty)$?

Inequalities. One place where interval notation often comes up is in solving inequalities. The solution to an inequality usually isn't a single number, but a set of numbers. We write those *solution sets* using intervals.

55. What is the solution set to the inequality $-3x + 1 < 5$?

56. What is the solution set to the inequality $2x^2 < 4$?

57. Let's solve the inequality $x^2 + 4x - 5 \geq 0$ in a step by step fashion.
    (a) Factor the left hand side of the inequality.

    (b) If x is less than -5…
        i. …is the first factor positive or negative?

        ii. …is the second factor positive or negative?

iii. …is the inequality true or false?

(c) If x is in the set $(-5,1)\ldots$

i. …is the first factor positive or negative?

ii. …is the second factor positive or negative?

iii. …is the inequality true or false?

(d) If x is in the set $(1,\infty)\ldots$

i. …is the first factor positive or negative?

ii. …is the second factor positive or negative?

iii. …is the inequality true or false?

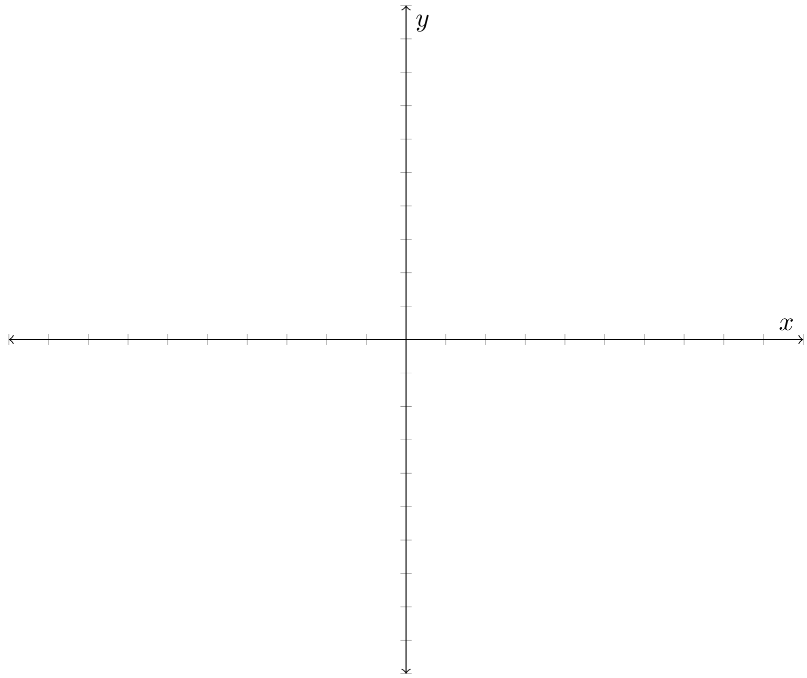
(e) Have we covered all possible values of x? If not, what's remaining, and is the inequality true or false for those values?

(f) What is the solution set to this inequality?

58. Can you solve $\dfrac{(x-2)(x+3)}{x} \le 0$?

Distance between points. We'll deal with points in the plane quite a bit.

59. Plot the points $(-2, 5)$ and $(-3, -1)$ in the axes below.



60. What does the Pythagorean Theorem say?

61. What is the distance between those two points?

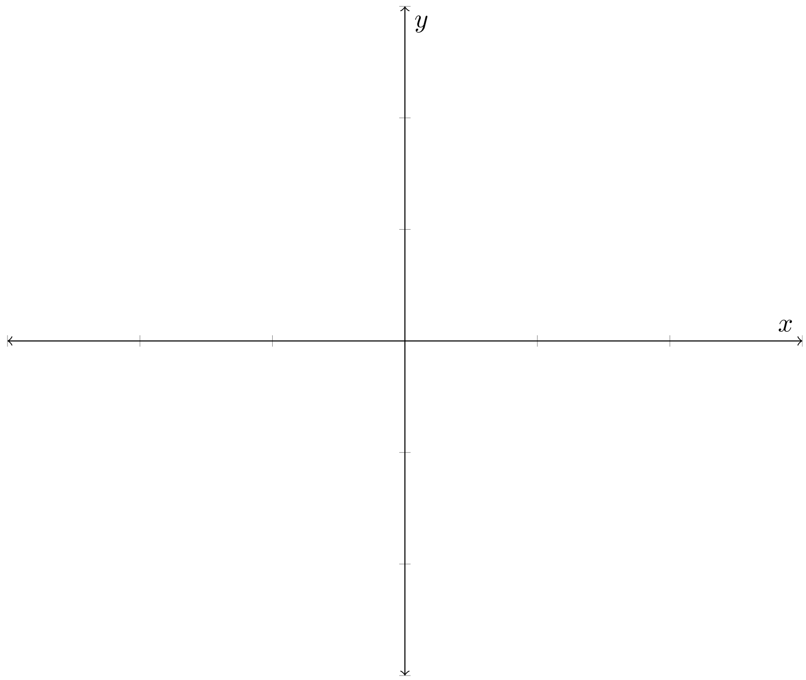
62. Can you write a formula for the distance between two points, (x_1, y_1) and (x_2, y_2)?

Circles. How can you describe a circle via an equation?

63. (a) On the axes below, wraw a circle of radius 2 centered at the origin.

    (b) Let's say the point (a, b) happens to lie on this circle. Pick a point on the circle. Any point. Draw a dot at that point.

    (c) Draw a line connecting the point (a, b) to the origin. Draw another line, perpendicular to the x-axis, that connects (a, b) and the x-axis. With those two lines and the x-axis, what kind of shape do you get?



(d) What are the lengths of the sides of this shape? Your answers may be in terms of a and b.

(e) Write down a relation between your three sides.

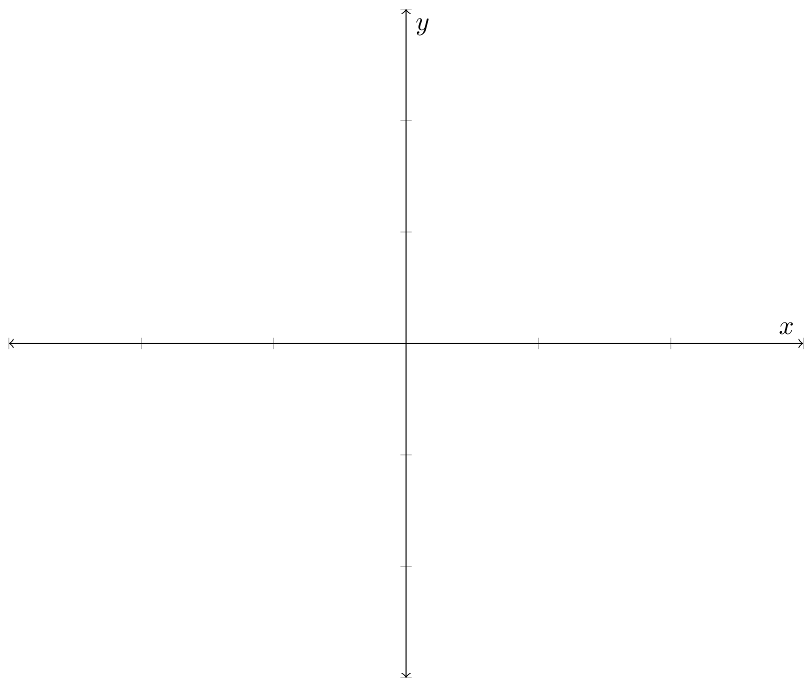
(f) Is that relation true for any point on the circle?

(g) Are there any points *not* on the circle for which that relation is true?

(h) What's the equation of this circle?

64. What would be the equation of a circle of radius 1 centered at the origin?

65. Take your equation and replace x by $x - 2$ (remember to use parentheses!). What shape does this new equation correspond to? Draw it. (If at first you don't succeed, plot some points, take a guess, see if your guess works...)



66. What would be the equation of a circle centered at $(3, -2)$ with radius 5?

Equations. Where a lot of this comes up – factoring, distributive law, rewriting expressions, etc. is in solving equations. You've done some of that already, now let's practice some more.

67. Solve $12x - (3 + 5x) = 10$.

68. Solve $\frac{x}{x+1} - \frac{x+1}{x} = 0$.

69. Solve for q: $\sqrt{1 - \frac{x}{3q}} = z$.

70. Where would the line $y = 2x + 1$ intersect the circle $(x - 1)^2 + y^2 - 4 = 0$? Draw a picture and find the exact solutions.