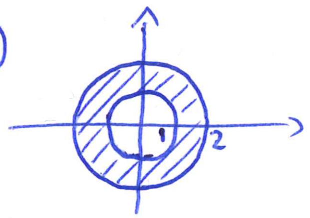
Double Integrals in Polar Coordinates

1)



$$\iint_D xy\, dA = \int_0^{\frac{\pi}{2}}\int_1^2 r^3\cos\theta\sin\theta\, dr\, d\theta$$

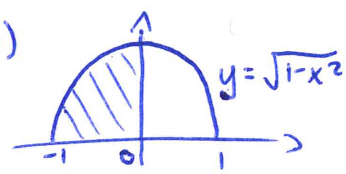
$$= \int_0^{\frac{\pi}{2}} \left[\tfrac{1}{4}r^4\cos\theta\sin\theta\right]_1^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} \tfrac{15}{4}\cos\theta\sin\theta\, d\theta \qquad \underline{\text{let } u=\sin\theta}$$

$$= \int_0^1 \tfrac{15}{4}u\, du$$

$$= \left[\tfrac{15}{8}u^2\right]_0^1 = \underline{\underline{\tfrac{15}{8}}}$$

2) (a)



$$\int_{\frac{\pi}{2}}^{\pi}\int_0^1 8r^5\cos^3\theta\sin\theta\, dr\, d\theta$$

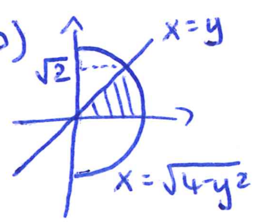
$$= \int_{\frac{\pi}{2}}^{\pi} \left[\tfrac{4}{3}r^6\cos^3\theta\sin\theta\right]_0^1 d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} \tfrac{4}{3}\cos^3\theta\sin\theta\, d\theta \qquad \underline{\text{let } u=\cos\theta}$$

$$= \int_0^{-1} -\tfrac{4}{3}u^3\, du$$

$$= \left[-\tfrac{1}{3}u^4\right]_0^{-1} = \underline{\underline{-\tfrac{1}{3}}}$$

(b)



$$\int_0^{\frac{\pi}{4}}\int_0^2 re^{r^2}\, dr\, d\theta \qquad \underline{\text{let } u=r^2}$$

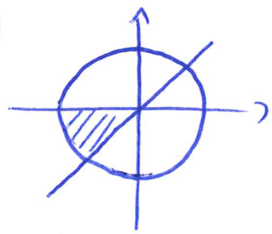
$$= \int_0^{\frac{\pi}{4}}\int_0^4 \tfrac{1}{2}e^u\, du\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[\tfrac{1}{2}e^u\right]_0^4 d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tfrac{1}{2}e^4 - \tfrac{1}{2}e\ d\theta$$

$$= \underline{\underline{\tfrac{\pi}{4}\left(\tfrac{1}{2}e^4 - \tfrac{1}{2}e\right)}}$$

3) (a)



(b) $\displaystyle\int_{\pi}^{\frac{5\pi}{4}}\int_0^4 \frac{3r^3\cos^2\theta}{r\sin\theta}\,dr\,d\theta$

4) (a) $\displaystyle\int_0^{2\pi}\int_0^2 r^2\cos\theta - r^2\sin\theta\,dr\,d\theta$

$\displaystyle= \int_0^{2\pi}\left[\tfrac{1}{3}r^3\cos\theta - \tfrac{1}{3}r^3\sin\theta\right]_0^2 d\theta$

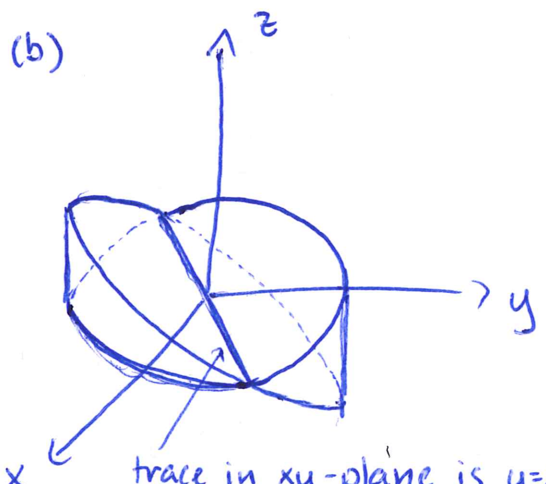
$\displaystyle= \int_0^{2\pi}\tfrac{8}{3}\cos\theta - \tfrac{8}{3}\sin\theta\,d\theta$

$\displaystyle= \left[\tfrac{8}{3}\sin\theta + \tfrac{8}{3}\cos\theta\right]_0^{2\pi}$

$= (0+\tfrac{8}{3}) - (0+\tfrac{8}{3}) = \underline{\underline{0}}$

Not the volume, positive and negative parts are equal

(b)



trace in xy-plane is $y=x$
(switches between positive/negative)

$\displaystyle V = 2\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\int_0^2 r^2\cos\theta - r^2\sin\theta\,dr\,d\theta$

$\displaystyle= 2\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\left[\tfrac{1}{3}r^3\cos\theta - \tfrac{1}{3}r^3\sin\theta\right]_0^2 d\theta$

$\displaystyle= 2\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\tfrac{8}{3}\cos\theta - \tfrac{8}{3}\sin\theta\,d\theta$

$\displaystyle= 2\left[\tfrac{8}{3}\sin\theta + \tfrac{8}{3}\cos\theta\right]_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}$

$\displaystyle= 2\left[\left(\tfrac{8}{3\sqrt{2}} + \tfrac{8}{3\sqrt{2}}\right) - \left(-\tfrac{8}{3\sqrt{2}} - \tfrac{8}{3\sqrt{2}}\right)\right]$