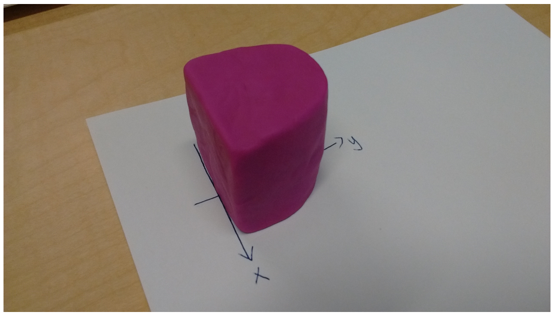
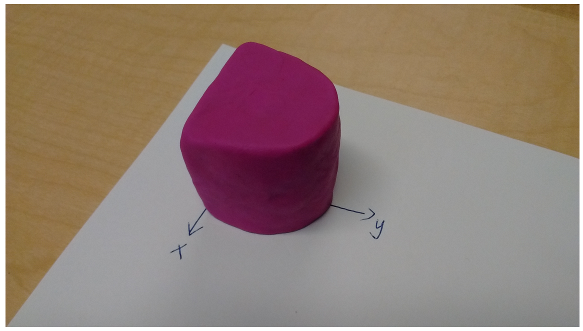
MATH 2110Q—Multivariable Calculus

MODELING SOLIDS

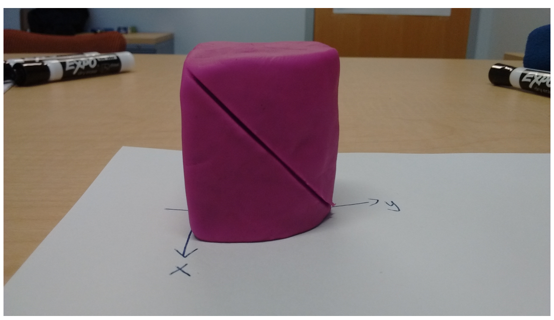
Let's use Play-Doh to model the solid bounded by $z = 0$, $y = 0$, $y = 4 - x^2$, and $z = 4 - y$, and represent its volume as a triple integral.

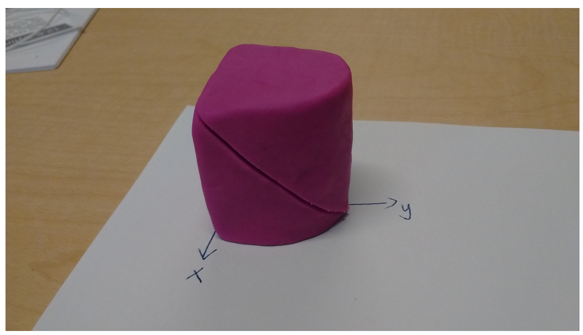
Begin by building the solid bounded by $z = 0$ (plane on bottom), $y = 0$ (plane on the left), $y = 4 - x^2$ (a parabolic cylinder that extends parallel to the z-axis making a tunnel-like shape), and $z = 4$ (this is arbitrarily chosen to give a bounding plane on top).



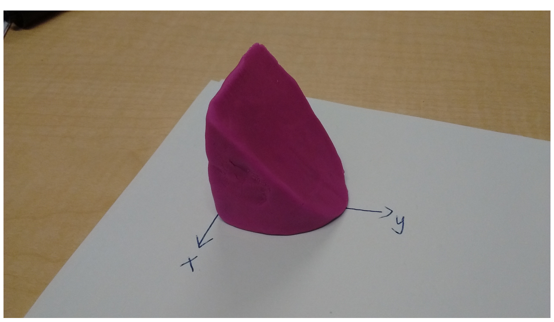


2) Now introduce the plane $z = 4 - y$ (a plane that, in the perspective shown, slices from the upper left to the lower right, with z-intercept 4) by slicing the solid along the line shown.

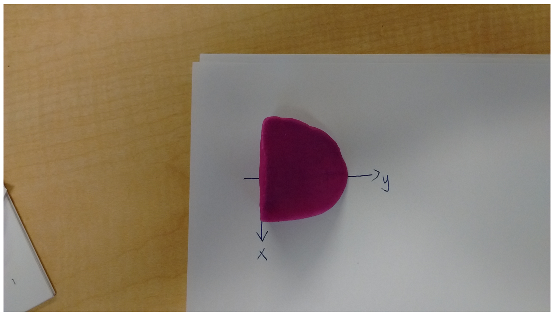




Removing the upper piece gives us our solid:



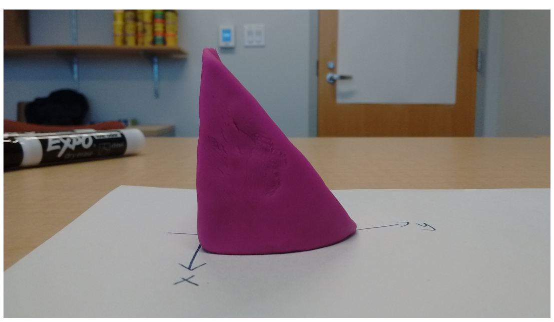
Let's now think a little about the shadows/silhouettes, which are key in setting up triple integrals over this solid region. Below, we see the view from the positive z-axis, looking directly down upon the solid.



In the picture above, we see that xy-shadow (or silhouette) that is bounded by $y = 0$ and $y = 4 - x^2$, with x going from -2 to 2. We also see the plane $z = 4 - y$ on top of the solid, with $z = 0$ on the bottom (it's type z). These bounds give the triple integral

$$V = \iiint\limits_E 1\, dV = \int_{-2}^{2} \int_0^{4-x^2} \int_0^{4-y} 1\, dz\, dy\, dx.$$

If we look at the solid from the positive x-axis, we see the triangular yz-shadow (or silhouette) that is bounded by $z = 0$ and $z = 4 - y$ with y going from 0 to 4. We also see the surface $x = \sqrt{4-y}$ on the front of the solid, with $x = -\sqrt{4-y}$ on the back (it's type x).



With these bounds, we get the triple integral $V = \displaystyle\int_0^4 \int_0^{4-y} \int_{-\sqrt{4-y}}^{\sqrt{4-y}} 1\, dx\, dz\, dy.$

If we look at the solid from the positive y-axis, we see the square xz-shadow (or silhouette) but we see why the solid is not type y: we see more than one bounding surface from this perspective as we can see the plane $z = 4 - y$ and we also see the surface $y = 4 - x^2$ (in the lower corners—not so easy to see here, sorry!).