Exam 3 Review T/F

1. T/F (with justification)

   If $\sum_{n=0}^{\infty} c_n$ converges then $\sum_{n=0}^{\infty} c_n x^n$ converges when $|x| < 1$.

   Solution: TRUE.

   Since $\sum_{n=0}^{\infty} c_n x^n$ is centered at 0, convergence at 1 implies it at x with $|x - 0| < |1| = 1$.

   Remark. The power series doesn't have to have radius of convergence 1; it could be larger.

2. T/F (with justification)

   If $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 3 then $\sum_{n=0}^{\infty} c_n x^{2n}$ has radius of convergence 9.

   Solution: FALSE.

   We are told $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < 3$ and diverges for $|x| > 3$, so $\sum_{n=0}^{\infty} c_n x^{2n} = \sum_{n=0}^{\infty} c_n (x^2)^n$ converges for $|x^2| < 3$ and diverges for $|x^2| > 3$, which is the same as convergence for $|x| < \sqrt{3}$ and divergence for $|x| > \sqrt{3}$, so the radius of convergence is $\sqrt{3}$, not 3.

3. T/F (with justification)

   If $f(x) = 1 + 3x - 2x^2 + 5x^3 + \cdots$ for $|x| < 1$ then $f'''(0) = 30$.

Solution: TRUE

The coefficient of x^3 is $f'''(0)/3! = f'''(0)/6$, so $f'''(0)/6 = 5$. Thus $f'''(0) = 30$.

4. T/F (with justification)

   The 2nd-degree Taylor polynomial at 0 for $\sqrt{1+x}$ is $1 + (1/2)x - (1/4)x^2$.

Solution: FALSE

If $f(x) = \sqrt{1+x}$ then $f'(x) = \dfrac{1}{2\sqrt{1+x}}$ and $f''(x) = -\dfrac{1}{4(1+x)^{3/2}}$, so

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2.$$

Although $f''(0) = -1/4$, the coefficient of x^2 is $f''(0)/2$, not $f''(0)$.

5. T/F (with justification): Doubling the radius of a sphere will double the surface area of the sphere.

Solution: FALSE. The surface area of a sphere radius r is $4\pi r^2$. If r is replaced with $2r$ then the surface area formula changes by a factor of $2^2 = 4$, not 2.

6. T/F (with justification)

   The parametric curve $(\sin t, -\cos t)$ as t increases traces out a circle counterclockwise.

Solution: TRUE.

This path traces out a unit circle at "unit" speed. At $t = 0$ the point is $(0, -1)$. At $t = \pi/2$, this point is $(1, 0)$. Going from the first point to the second at unit speed is counterclockwise missions.

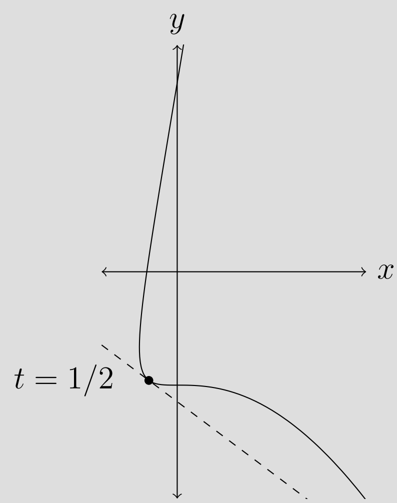
7. T/F (with justification)

   On the parametric curve $(x, y) = (t^2 - 2t, t^3 - 3)$ the graph is increasing at the point where $t = 1/2$.

Solution: FALSE
We have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t-2} = \frac{3t^2}{2(t-1)}.$$

At $t = 1/2$ (the point is $(x, y) = (-.75, -2.875)$) the derivative is $(3/4)/(2(1/2-1)) = -3/4 < 0$.
Remark. A graph of the parametric curve is shown below, with the point at $t = 1/2$ and the tangent line indicated. The derivative (slope of tangent line) is visibly negative.



T/F (with justification)

Every point in the plane besides the origin can be written in polar coordinates (r, θ) with $r < 0$.

Solution: TRUE.

Every point besides the origin has polar coordinates (r, θ) with $r > 0$. The polar coordinates $(-r, \theta+\pi)$ describe the same point: its Cartesian coordinates are

$$(-r)\cos(\theta+\pi) = (-r)(-\cos\theta) = r\cos\theta \text{ and } (-r)\sin(\theta+\pi) = (-r)(-\sin\theta) = r\sin\theta.$$

8. The fifth degree Taylor polynomial for $f(x) = \sin x$ centered at $a = 0$ is $1 + \frac{x^3}{3!} +$

$\frac{x^5}{5!}$.

Solution: FALSE.

As $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, the fifth degree Taylor polynomial for $f(x) = \sin x$ centered at $a = 0$ should be $x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

If you cannot memorize the formula, here is the calculation. For $f(x) = \sin x$, we have

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0
5	$\cos x$	1

So

$$\begin{aligned} T_5(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!}. \end{aligned}$$

9. If the power series $\sum_{k=0}^{\infty} a_k (x-4)^k$ has a radius of convergence equal to 2 then $\sum_{k=0}^{\infty} a_k$ diverges.

Solution: FALSE.

The power series $\sum_{k=0}^{\infty} a_k (x-4)^k$ centered at $a = 4$ and has radius $R = 2$ from the statement. Since $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} a_k \cdot 1^k = \sum_{k=0}^{\infty} a_k \cdot (5-4)^k$, it is the power series at $x = 5$. As $|5 - 4| = 1 < 2$, that is $|x - a| < R$, so the power series converges at $x = 5$, then we have the convergence of $\sum_{k=0}^{\infty} a_k$.

10. Suppose the power series $\sum_{k=1}^{\infty} c_k(x-3)^k$ diverges at $x = 0$ and converges at $x = 5$. Which of the following are possible?

a) The power series converges when $x = 1$.
b) The power series diverges when $x = 2$.
c) The radius of convergence is 3.

- ◯ 1) a and b
- ◯ 2) b and c
- ◯ 3) a and c
- ◯ 4) just one of a, b, or c
- ◯ 5) a, b, and c

Solution: 3) a and c.

Since $\sum_{k=1}^{\infty} c_k(x-3)^k$ is centered at $a = 3$ and it converges at $x = 5$, the power series has a radius of convergence $R \geq |5-3| = 2$. At the same time, the power series is centered at $a = 3$ and it diverges at $x = 0$, so the power series has radius of convergence $R \leq |0 - 3| = 3$. Therefore, $2 \leq R \leq 3$.

For $x = 1$, $|1 - 3| = 2$, so if R is any larger value than 2, say $R = 2.5$, the power series converges. If $R = 2$, then $x = 1$ is on an endpoint of the interval and *could* converge. The statement is possible.

For $x = 2$, $|2 - 3| = 1$, since $1 < 2 \leq R$, $|x - a|$ is always less than R, so the power series always converges for any $2 \leq R \leq 3$. The statement is impossible.

Since $2 \leq R \leq 3$, so it's possible that $R = 3$.