Exam 2 Review T/F

1. T/F (with justification): The work required to stretch a spring having spring constant k a distance x from its equilibrium (rest) position is kx.

   Solution: FALSE. The expression kx is the *force* needed to hold the spring at a distance x from equilibrium. The work needed to stretch it to a distance x is $\int_0^x ky\,dy = kx^2/2$.

2. T/F (with justification): Every bounded sequence is convergent.

   Solution: FALSE.

   Try $a_n = (-1)^n$.

3. T/F (with justification): If $a_n \to 0$ as $n \to \infty$ then the series $\sum_{n=1}^{\infty} a_n$ converges.

   Solution: FALSE.

   A counterexample is the harmonic series.

4. T/F (with justification): The convergence of a series $\sum_{n=1}^{\infty} a_n$ is unaffected by dropping its first few terms.

   Solution: TRUE.

   If $\sum_{n=1}^{N} a_n \to S$ as $N \to \infty$ then $\sum_{n=2}^{N} a_n \to S - a_1$, $\sum_{n=3}^{N} a_n \to S - a_1 - a_2$, and so on.

   Conversely, if the series with a few initial terms removed converges to some definite value L, then adding the initial terms back in will give a series whose value is L plus those initial few terms.

5. T/F (with justification): If $f(x)$ is continuous, positive, and decreasing for $x \geq 1$, and $\int_1^\infty f(x)\,dx$ converges then $\sum_{n=1}^\infty f(n) = \int_1^\infty f(x)\,dx$.

Solution: FALSE.

Try a geometric series $\sum_{n=1}^\infty \frac{1}{2^n}$, which equals $\frac{1/2}{1-1/2} = 1$ while $\int_1^\infty \frac{dx}{2^x} = \frac{1}{2^x(-\ln 2)}\Big|_1^\infty = \frac{1}{2\ln 2} \approx .72.$

6. T/F (with justification)

The divergence of p-series $\sum_{n=1}^\infty \frac{1}{n^p}$ for $0 < p < 1$ follows from divergence of the harmonic series $\sum_{n=1}^\infty \frac{1}{n}$ by the comparison test.

Solution: TRUE.

For $0 < p < 1$ we have $\frac{1}{n^p} > \frac{1}{n}$, so the divergence of $\sum_{n=1}^\infty \frac{1}{n}$ implies divergence of $\sum_{n=1}^\infty \frac{1}{n^p}$.

7. T/F (with justification)

The convergence of p-series $\sum_{n=1}^\infty \frac{1}{n^p}$ for $p > 1$ follows from divergence of the harmonic series $\sum_{n=1}^\infty \frac{1}{n}$ by the comparison test.

Solution: FALSE.

That is, this *reasoning* is false even though certainly the premise that p-series converge for $p > 1$ is true. The comparison test never lets us argue that one series converges because another series diverges. It only lets us deduce convergence of one series from convergence of another series and divergence of one series from divergence of another series.

8. T/F (with justification)

   The infinite series $\sum_{n=1}^{\infty} \frac{\cos n}{n}$ is alternating.

   Solution: FALSE.

   The numbers $\cos 1, \cos 2, \cos 3, \ldots$ have successive signs $+, -, -, -, +, +, +$, which is not alternating (apparent even from first three terms).

9. T/F (with justification)

   For the alternating series $s = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$, s lies between s_{100} and s_{101}.

   Solution: TRUE.

   The reason is that every alternating series whose terms in absolute value satisfy $b_{n+1} < b_n$ lies in between consecutive partial sums. Thus s is between s_{100} and s_{101}.

   Remark. For the series s, its first term is positive so $s_{100} < s < s_{101}$.

10. T/F (with justification)

    Convergence of a p-series for $p > 1$ can be shown with the ratio test.

    Solution: FALSE.

    For a p-series, $a_n = \frac{1}{n^p}$ and $\frac{a_{n+1}}{a_n} = a_{n+1} \cdot \frac{1}{a_n} = \frac{1}{(n+1)^p} n^p = \frac{n^p}{(n+1)^p} = \left(\frac{n}{n+1}\right)^p$. As $n \to \infty$ we have $n/(n+1) \to 1$, so $a_{n+1}/a_n \to 1$. Thus the ratio test is inconclusive.

11. T/F (with justification)

    There is an infinite series whose terms can be rearranged to be an infinite series with a different value.

Solution: TRUE.

The terms in the alternating harmonic series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

can be rearranged to get a series with a different value. See p. 742 of the textbook for one example: two positive terms followed by a negative term is a series equal to $(3/2)S$. Here is another rearrangement. Ordering one positive followed by two negatives gives

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \cdots,$$

which is the same as

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \cdots = \frac{1}{2}S.$$

12. The geometric series $\displaystyle\sum_{n=4}^{\infty} \left(\frac{1}{3}\right)^n$ converges to $\dfrac{3}{2}$.

Solution: FALSE.

Since $\left|\frac{1}{3}\right| < 1$, $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-1/3} = \frac{3}{2}$, so

$$\begin{aligned}\sum_{n=4}^{\infty} \left(\frac{1}{3}\right)^n &= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n - \left(\frac{1}{3}\right)^0 - \left(\frac{1}{3}\right)^1 - \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^3 \\ &= \frac{3}{2} - 1 - \frac{1}{3} - \frac{1}{9} - \frac{1}{27} \\ &\neq \frac{3}{2}.\end{aligned}$$

13. The series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$ converges conditionally.

Solution: FALSE.

As $\sum_{k=1}^{\infty}\left|\frac{(-1)^k}{k^3}\right| = \sum_{k=1}^{\infty}\frac{1}{k^3}$ is a p-series with $p = 3 > 1$, it is convergent. Therefore, by definition, $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}$ is absolutely convergent.

14. If $\sum_{n=1}^{\infty}|a_n|$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

Solution: FALSE.

Any conditionally convergent series is a counterexample, by definition, $\sum_{n=1}^{\infty} a_n$ converges while $\sum_{n=1}^{\infty}|a_n|$ diverges.

For one particular counterexample, $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges, but $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right|$ converges.