8.1 Arc Length

The Arc Length Formula. If f' is continuous on $[a,b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

If $x = g(y)$, $c \leq y \leq d$, and $g'(y)$ is continuous, then

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

1. **Example:** Write out the arc length of $y = \sqrt{x}$ on $1 \leq x \leq 2$ as a definite integral with respect to x.

   Thinking about the problem:

   Which formula should I use to determine the arc length and why? Have I seen a problem similar to this one before? If so, which formula did I use?

   I note that the formula to find arc length is $L = \displaystyle\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$, so I will need to find $f'(x)$ and determine if $f'(x)$ is continuous on $[1, 2]$.

   Doing the problem:

   The problem asks for the definite integral to find the arc length of $y = \sqrt{x}$ on $1 \leq x \leq 2$. I know that $f'(x) = \frac{1}{2}x^{-1/2}$, which is continuous on $[1, 2]$ and so the arc length is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_1^2 \sqrt{1 + \left(\frac{1}{2x^{1/2}}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{4x}}\, dx.$$

Solutions should show all of your work, not just a single final answer.

2. Write out the arc length of $x = y^2$ on $1 \le x \le 2$ as a definite integral with respect to y.

3. (a) Write out the arc length of $y = x^3$ for $0 \le x \le 2$ as a definite integral with respect to x.

   (b) Write out the arc length of $y = x^3$ for $0 \le x \le 2$ as a definite integral with respect to y.

   (c) Why is the integral in part (b) improper?

4. T/F (with justification): The arc length of the graph of $y = \sin x$ for $0 \le x \le \pi/2$ equals $\int_0^{\pi/2} \sqrt{1 + \sin^2 x}\, dx$.