Math 1132 Worksheet 7.8 Name: ______________________ Discussion Section: _____

7.8 Improper Integrals

Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

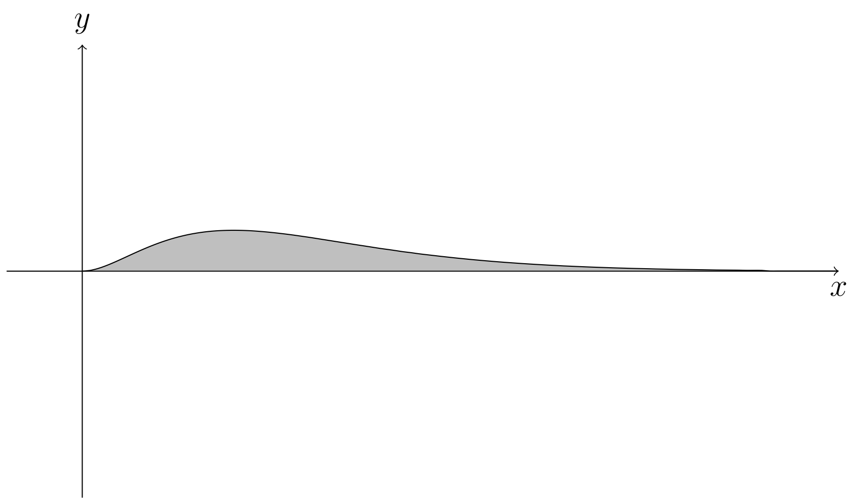
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

1. **Example:** Determine if $\int_0^{\infty} x^2 e^{-x}\,dx$ is convergent or divergent. If it is convergent, evaluate it.

Thinking about the problem:

What technique of integration should I use to evaluate the integral and why? Have I seen a problem similar to this one before? If so which technique did I use?

I know that I need to evaluate a definite integral, so I know my answer will be finite if the integral is convergent or the result will be that the integral diverges. First I consider the integrand, i.e., x^2e^{-x}. The region I am integrating over looks like this:



I notice that x^2e^{-x} is not defined at ∞, so I will need to take a limit of an improper integral, from 0 to infinity. Thus

$$\int_0^{\infty} x^2 e^{-x} = \lim_{a\to\infty} \int_0^{a} x^2 e^{-x}\,dx.$$

I can evaluate this integral using Integration By Parts. Then I should get my answer by taking the limit of the expression that I get after integration.

Doing the problem:

I see that

$$\int_0^\infty x^2 e^{-x}\,dx = \lim_{a\to\infty}\int_0^a x^2 e^{-x}\,dx.$$

By the Integration By Parts technique, I can fill in the table

$u = x^2$	$dv = e^{-x}$
$du = 2x$	$v = -e^{-x}$

and so

$$\int_0^a x^2 e^{-x}\,dx = -x^2 e^{-x}\Big|_0^a - \int_0^a (-e^{-x})(2x)\,dx$$

$$= -x^2 e^{-x}\Big|_0^a + \int_0^a 2xe^{-x}\,dx.$$

I will use the Integration By Parts technique again, so I fill in the table

$u = 2x$	$dv = e^{-x}$
$du = 2$	$v = -e^{-x}$

So

$$= -x^2 e^{-x}\Big|_0^a + \left(-2xe^{-x}\Big|_0^a - \int_0^a (-e^{-x})\cdot 2\,dx\right)$$

$$= -x^2 e^{-x}\Big|_0^a - 2xe^{-x}\Big|_0^a + \int_0^a 2e^{-x}\,dx$$

$$= -x^2 e^{-x}\Big|_0^a - 2xe^{-x}\Big|_0^a - 2e^{-x}\Big|_0^a$$

$$= (-a^2e^{-a} - 0) - (2ae^{-a} - 0) - (2e^{-a} - 2)$$

$$= -a^2e^{-a} - 2ae^{-a} - 2e^{-a} + 2.$$

So I find

$$\begin{aligned}\int_0^\infty x^2e^{-x}\,dx &= \lim_{a\to\infty}\int_0^a x^2e^{-x}\,dx\\ &= \lim_{a\to\infty} -a^2e^{-a} - 2ae^{-a} - 2e^{-a} + 2\\ &= \lim_{a\to\infty} -a^2e^{-a} - \lim_{a\to\infty} 2ae^{-a} - \lim_{a\to\infty} 2e^{-a} + \lim_{a\to\infty} 2.\end{aligned}$$

By L'Hospital's rule, $\lim_{a\to\infty} -a^2e^{-a} = \lim_{a\to\infty} \frac{-a^2}{e^a} = \lim_{a\to\infty} \frac{-2a}{e^a} = \lim_{a\to\infty} \frac{-2}{e^a} = 0$. Similarly, I can find $\lim_{a\to\infty} 2ae^{-a} = 0$. By taking the limit directly, I find $\lim_{a\to\infty} 2e^{-a} = 0$ and $\lim_{a\to\infty} 2 = 2$.

Therefore

$$\int_0^a x^2e^{-x}\,dx = 0 + 0 + 0 + 2 = 2.$$

Solutions should show all of your work, not just a single final answer.

2. Determine if $\int_1^\infty \frac{\cos x}{x^2} dx$ is convergent or divergent.

   (a) Draw the graph of the function $\frac{\cos x}{x^2}$.

   (b) Shade the region from your graph in part (a) that is to be integrated.

   (c) Is this function bounded above? Is this function bounded below?

3. Decide if $\int_0^{\infty} \frac{x}{x^2+1}\, dx$ is convergent or divergent. If it is convergent, evaluate it.

(a) Draw the graph of the function $\frac{x}{x^2+1}$.

(b) Shade the region from your graph in part (a) that is to be integrated.

(c) Evaluate the integral to determine if it is convergent or divergent.

4. Is $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)}$ convergent?

5. T/F (with justification) The integral $\displaystyle\int_0^2 \frac{dx}{x-1}$ is convergent.