7.7 Approximate Integration

Midpoint Rule:

$$\int_a^b f(x)dx \approx M_n = \Delta x[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{ midpoint of } [x_{i-1}, x_i].$$

For E_M the error in the Midpoint Rule, then

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

where $|f''(x)| \leq K$ for $a \leq x \leq b$.

Trapezoidal Rule:

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$x_i = a + i\Delta x.$$

For E_T the error in the Trapezoidal Rule, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

where $|f''(x)| \leq K$ for $a \leq x \leq b$.

Simpson's Rule:

$$\int_a^b f(x)dx \approx S_n$$

where

$$S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

and n is even and

$$\Delta x = \frac{b-a}{n}.$$

For E_S the error bound for Simpson's Rule, then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

where $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$.

1. **Example:**

   (a) Apply the Trapezoidal Rule to $\int_1^3 \sqrt{x}\,dx$ using $n = 4$ subintervals, rounding your approximation to 5 digits after the decimal point.

   (b) Use the error bound for the Trapezoidal Rule to determine an n such that the Trapezoidal Rule is *guaranteed* by the error bound to be with .01 of the value of the integral.

   Thinking about the problem:

   This problem requires me to apply the Trapezoidal Rule, so I need to compute the function in my integrand, i.e. $\sqrt{x}$ using $n = 4$ subintervals within the interval $[1, 3]$. Next, I can compute Δx (i.e., $\Delta x = \frac{b-a}{n}$) which will allow me to find the endpoints of the trapezoids under the curve $f(x) = \sqrt{x}$. Finally, I can use the Trapezoidal Rule to find an approximation to $\int_1^3 \sqrt{x}\,dx$.

   In part (b), I should recall the error bound for the Trapezoidal Rule. I know my error $|E_T|$ should be less than .01, and I know that $|E_T| \leq \frac{K(b-a)^3}{12n^2}$, so to ensure $|E_T| < .01$, I will solve n such that $\frac{K(b-a)^3}{12n^2} \leq .01$. Part (b) should be concluded by algebraically solving the inequality for n. I take note that the Trapezoidal Rule should be more accurate for larger n, so my answer should be n is greater or equal to a number. If my inequality is reversed, I may have made a mistake somewhere in my algebra or in my statement of the problem.

Doing the Problem:

In part (a), I see that I need to find Δx and x_0, x_1, x_2, x_3, and x_4 before I can apply the rule. I see that $\Delta x = \dfrac{b-a}{n}$, so $\Delta x = \dfrac{3-1}{4} = \dfrac{1}{2}$. By applying the formula $x_i = a + i\Delta x$, I find

x_0	1
x_1	1.5
x_2	2
x_3	2.5
x_4	3

so I can also find

$f(1)$	1
$f(1.5)$	1.2247...
$f(2)$	1.4142...
$f(2.5)$	1.5811...
$f(3)$	1.7320...

Thus I put my information into the Trapezoid Rule to find an approximation to $\int_1^3 \sqrt{x}\,dx$:

$$\int_1^3 \sqrt{x}\,dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

$$= \frac{.5}{2}[f(1) + 2f(1.5) + 2f(1) + 2f(2.5) + f(3)]$$

$$\approx 2.79306$$

This is the answer to part (a).

To find an n such that the error bound is less than .01, I will solve

$$\frac{K(b-a)^3}{12n^2} \leq .01.$$

In this case, I need to find K where $|f''(x)| \leq K$ for all $a \leq x \leq b$. Since I know $f(x) = \sqrt{x}$, I find that $f''(x) = -\frac{1}{4}x^{-3/2}$. For all values $1 \leq x \leq 3$, we know that $x^{3/2} \geq 1$, so $x^{-3/2} \leq 1$ and $|f''(x)| = \frac{1}{4}x^{-3/2} \leq \frac{1}{4}$. So I plug in K, b, and a into the inequality to find

$$\frac{\frac{1}{4}(3-1)^3}{12n^2} \leq .01$$

$$\frac{8}{48n^2} \leq .01$$

$$\frac{1}{6n^2} \leq .01$$

$$1 \leq .06n^2$$

$$\frac{1}{.06} \leq n^2$$

$$\sqrt{\frac{1}{.06}} \approx 4.082 \leq n.$$

However, we know that n must be a positive integer, so

$$5 \leq n$$

So, for any $n \geq 5$ (n is the number of subintervals) an approximation of $\int_1^3 \sqrt{x}\,dx$ using the Trapezoidal Rule will be within 0.01 of the actual value.

Solutions should show all of your work, not just a single final answer.

2. (a) Apply the Trapezoidal Rule to $\int_2^3 e^{x^2}\,dx$ using $n = 4$ subintervals, rounding your approximation to 5 digits after the decimal point.

(b) Find the error bound for the Trapezoidal Rule applied to $\int_2^3 e^{x^2}\,dx$ using $n = 4$ subintervals.

(c) Set up the error bound for the Trapezoidal Rule applied to $\int_2^3 e^{x^2}\,dx$ using general n subintervals.

(d) Use the error bound for the Trapezoidal Rule to determine an n such that the Trapezoidal Rule is *guaranteed* by the error bound to be with .01 of the value of the integral.

3. (a) Apply Simpson's Rule to $\int_1^2 \sqrt{x}\, dx$ using $n = 4$ subintervals, rounding your approximation to 5 digits after the decimal point.

(b) Use the error bound for Simpson's Rule to determine an n such that Simpson's Rule is *guaranteed* by the error bound to be with 10^{-6} of the value of the integral. (Remember n must be even.)

4. T/F (with justification) The Trapezoidal Rule for $\int_a^b f(x)\, dx$ has no error if $f(x)$ is linear.