7.4 Integration by Partial Fractions

Remember:

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C.$$

Note:

$$\frac{1}{x(x^2 + a)} = \frac{A}{x} + \frac{Bx + C}{x^2 + a}$$

and

$$\frac{1}{x(x + b)^2} = \frac{A}{x} + \frac{B}{x + b} + \frac{C}{(x + b)^2}$$

1. **Example**. Evaluate $\displaystyle\int \frac{2x + 1}{x^2 - 4}\, dx$.

   Thinking about the problem:

   Which technique of integration should I use to evaluate the integral and why? Have I seen a problem similar to this one before? If so, which technique did I use?

   I know that I need to evaluate an indefinite integral, so I know my answer will include $+C$. To determine which technique I should use, I will focus on the integrand, i.e., $\displaystyle\frac{2x + 1}{x^2 - 4}$. It does not appear as if I could use the Substitution technique nor the Integration by Parts technique. I might be able to use the Trig Substitution technique, but it does not appear necessary in this case. Therefore, I will use the Integration by Partial Fractions technique. I see that I can factor the denominator $(x^2 - 4)$ into a product of linear factors, that is, $x^2 - 4 = (x + 2)(x - 2)$. So, by the Integration by Partial Fractions technique, I will simplify the integrand $\frac{2x+1}{x^2-4}$ into $\frac{A}{x+2}+\frac{B}{x-2}$ where A and B are constants. At this point, I can solve for A and B. Finally, since $\int \frac{2x+1}{x^2-4}\,dx = \int \frac{A}{x+2}\,dx + \int \frac{B}{x-2}\,dx$, I can solve the integral.

Doing the Problem:

The problem asks me to evaluate the integral $\int \frac{2x+1}{x^2-4}\,dx$. I choose to evaluate the integral using the technique of Integration by Partial Fractions. I can factor the denominator into the factors $x^2-4=(x+2)(x-2)$, so I can simplify the integrand $\frac{2x+1}{x^2-4}=\frac{A}{x+2}+\frac{B}{x-2}$. Next, I solve for A and B by multiplying both sides by the denominator x^2-4. We find

$$2x+1 = A(x-2)+B(x+2).$$

Setting $x=2$, we find

$$\begin{aligned} 2(2)+1 = 5 = A(0)+B(2+2) &= 4B \\ 5 &= 4B \\ \frac{5}{4} &= B. \end{aligned}$$

Setting $x=-2$, we find

$$\begin{aligned} 2(-2)+1 = -3 = A(-2-2)+B(0) &= -4A \\ -3 &= -4A \\ \frac{3}{4} &= A. \end{aligned}$$

Therefore, we see

$$2x + 1 = \frac{3}{4}(x - 2) + \frac{5}{4}(x + 2)$$

and

$$\begin{aligned}
\int \frac{2x+1}{(x+2)(x-2)}\,dx &= \int \frac{3}{4}\cdot\frac{1}{x+2} + \frac{5}{4}\cdot\frac{1}{x-2}\,dx \\
&= \frac{3}{4}\int \frac{1}{x+2}\,dx + \frac{5}{4}\int \frac{1}{x-2}\,dx \\
&= \frac{3}{4}\ln(|x+2|) + \frac{5}{4}\ln(|x-2|) + C.
\end{aligned}$$

Solutions should show all of your work, not just a single final answer.

2. Evaluate $\displaystyle\int \frac{dx}{x^2 - 4x + 3}$.

   (a) State the technique of integration you would use in this problem.

   (b) If you choose to use the Partial Fractions technique, rewrite the integrand as two fractions with A and B.

   (c) Find coefficients A and B in part (b).

3. Evaluate $\int \frac{x^2 + x + 1}{x(x^2 + 4)}\, dx$.

(a) If you choose to use the Partial Fractions technique, how would you rewrite the integrand?

(b) Find coefficients for the new integral found using the Partial Fractions technique.

(c) Evaluate the new integral found using the Partial Fractions technique.

4. Evaluate $\displaystyle\int \frac{x+1}{x^3 - 8x^2 + 16x}\,dx$.

5. T/F (with justification) Computing $\displaystyle\int \frac{x}{x^2-1}\,dx$ requires partial fractions.