7.3 Trigonometric Substitution

In each of the following trigonometric substitution problems, draw a triangle and label an angle and all three sides corresponding to the trigonometric substitution you select.

**Table of Trigonometric Substitution**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Substitution</th>
<th>Identity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{a^2 - x^2}$</td>
<td>$x = a \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$</td>
<td>$1 - \sin^2 \theta = \cos^2 \theta$</td>
</tr>
<tr>
<td>$\sqrt{a^2 + x^2}$</td>
<td>$x = a \tan \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$</td>
<td>$1 + \tan^2 \theta = \sec^2 \theta$</td>
</tr>
<tr>
<td>$\sqrt{x^2 - a^2}$</td>
<td>$x = a \sec \theta$, $0 \leq \theta &lt; \frac{\pi}{2}$ or $\pi \leq \theta &lt; \frac{3\pi}{2}$</td>
<td>$1 + \tan^2 \theta = \sec^2 \theta$</td>
</tr>
</tbody>
</table>

1. **Example:** Evaluate $\int \frac{dx}{\sqrt{9 - x^2}}$.

*Thinking about the problem:*

What technique of integration should I use to evaluate the integral and why? Have I seen a problem similar to this one before? If so, which technique did I use?

I know that I need to evaluate an indefinite integral, so I know my answer will include $+C$. To determine which technique to use, I will focus on the integrand, i.e., $\frac{1}{\sqrt{9 - x^2}}$.

I think I will use the technique of Trigonometric Substitution to evaluate this integral. I noticed that the denominator of the integrand is $\sqrt{9 - x^2}$, which is a form found in the Table of Trigonometric Substitution. I look at the table and find that the substitution
I want to use is \( x = 3 \sin \theta \). I can then simplify my integral with this substitution and integrate.

**Doing the problem:**

I will evaluate \( \int \frac{dx}{\sqrt{9 - x^2}} \) using the technique of Trigonometric Substitution. I will let \( x = 3 \sin \theta \) and \( dx = 3 \cos \theta \, d\theta \). Then I can draw a triangle using my choice of substitution and find the following picture:

![Diagram showing a triangle with sides labeled x, 3, \sqrt{9 - x^2}, and \theta.

So the new integral is

\[
\int \frac{dx}{\sqrt{9 - x^2}} = \int \frac{3 \cos \theta \, d\theta}{\sqrt{9 - (3 \sin \theta)^2}}
\]

\[
= \int \frac{3 \cos \theta \, d\theta}{\sqrt{9(1 - \sin^2 \theta)}}
\]

\[
= \int \frac{3 \cos \theta \, d\theta}{\sqrt{9 \cos^2 \theta}}
\]

\[
= \int \frac{3 \cos \theta \, d\theta}{3 \cos \theta}
\]

\[
= \int d\theta
\]

\[
= \theta + C.
\]
Since the substitution we used was \( x = 3 \sin \theta \), then \( \theta = \sin^{-1}\left(\frac{x}{3}\right) \). So

\[
\int \frac{dx}{\sqrt{9 - x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C.
\]

Solutions should show all of your work, not just a single final answer.

2. Evaluate \( \int \frac{dx}{(9 + x^2)^{3/2}} \).

(a) State the technique of integration you would use to evaluate the integral.

(b) Which substitution would you use for \( x \)? What would \( dx \) be?

(c) Based on the choice for \( x \), fill in the triangle:

(d) Using (b), write the new integral.
3. Evaluate \( \int \frac{\sqrt{x^2 - 9}}{x^3} \, dx \).

(a) State the technique of integration you would use to evaluate the integral.

(b) Which substitution would you use for \( x \)? What would \( dx \) be?

(c) Based on the choice for \( x \), fill in the triangle:

\[
\theta
\]

(d) Using (b), write the new integral.

(e) Using (c) and (d), evaluate the integral.
4. Evaluate \[ \int_{0}^{3} \frac{x^2}{\sqrt{9 - x^2}} \, dx. \] (Hint: When you make a trigonometric substitution, include the bounds of integration in the substitution.)

5. T/F (with justification): To evaluate \[ \int \frac{dx}{x^2 \sqrt{x^2 + 2}} \] by trigonometric substitution, use \[ x = 2 \tan \theta. \]