Math 1132 Worksheet 6.4 Name: ________________________ Discussion Section: _____

6.4 Work

Force formula for springs. By Hooke's Law, the force required to maintain a spring stretched x units beyond its natural length is $f(x) = kx$ where k is a positive constant called the spring constant.

Work. The work done in moving an object from a to b is

$$W = Fd$$

if the force begin applied is constant.

The work done in moving an object from a to b is

$$W = \int_a^b f(x)dx$$

if the force begin applied is variable.

Work formula for tanks. The integral for work required to empty a tank of liquid from height a to b is

$$\int_a^b \rho \cdot g \cdot D(y) \cdot A(y)\, dy$$

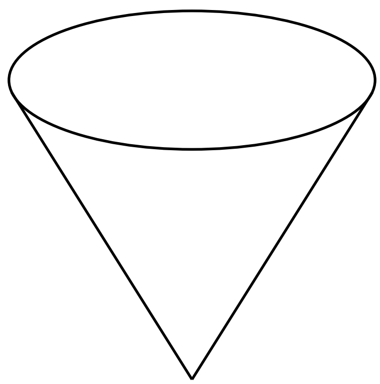
where $D(y)$ is the depth of a Δy slice, $A(y)$ is the area of the cross section at y, ρ is the density of the liquid, g is the acceleration of the liquid (e.g., gravity), and a and b represent the water levels in the tank.

1. **Example:** An inverted conical tank with radius 6 ft and height 20 ft is full of water. Find the work required to pump the water out of the top of the tank. Let $y = 0$ at the bottom of the tank. Use the fact that water weighs 62.5 lb/ft^2.

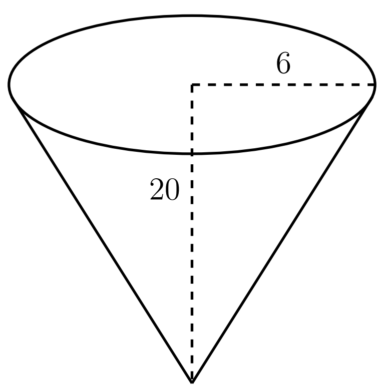
   Thinking about the problem:

   How should I approach this problem? Have I seen a problem similar to it before? If so, how did I approach it?

   First, I know what an inverted conical tank would look like, so I draw it.

   Then I can label the radius and the height as stated in the problem.

   Now I know what my tank looks like. Using the dimensions given in the problem, I will need to find a, b, ρ, g, $D(y)$, and $A(y)$ as the integral for work required to empty a tank is

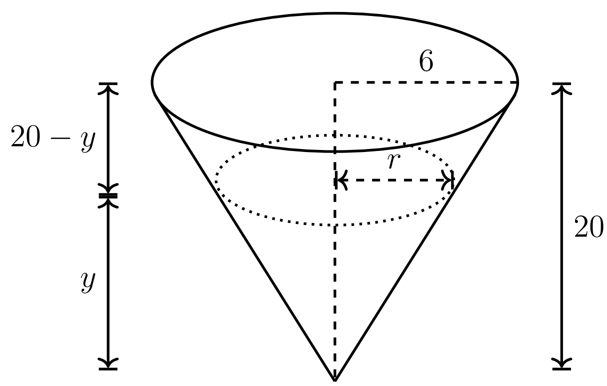
$$\int_a^b \rho \cdot g \cdot D(y) \cdot A(y)\, dy.$$

Doing the problem:

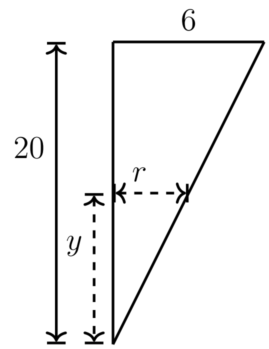
The problem asks me for the work required to pump water out of a tank. So I start with the formula

$$\int_a^b \rho \cdot g \cdot D(y) \cdot A(y)\, dy.$$

Since water weighs 62.5 lb/ft^2, I know that $\rho \cdot g = 62.5$ (the gravity is incorporated in the unit lb). I can situate the cone such that $y = 0$ is at the bottom of the tank, so I will label my cone as follows:



I know that I will empty the entire tank, so I know that I will integrate from the bottom of the tank ($y = 0$) to the top ($y = 20$). So I deduce that $a = 0$ and $b = 20$. Let y be the height of the slice so my $D(y) = 20 - y$. To find $A(y)$, I will need to find the area of the slice. I know that area of a circle is πr^2, so I need to find r as a function of y. However, I notice that this can be considered as a problem with similar triangles, so I draw the following figure.



By law of similar triangles, I know that

$$\frac{r}{y} = \frac{6}{20}$$

$$r = y \cdot \frac{3}{10}.$$

Therefore the area of the slice

$$\begin{aligned} A(y) &= \pi r^2 \\ &= \pi \left(y \cdot \frac{3}{10} \right)^2 \\ &= \pi y^2 \cdot \frac{9}{100}. \end{aligned}$$

Now I have found all the parts of my integral so I find that the work to pump the water out of the top of the tank is

$$
\begin{aligned}
W &= \int_a^b \rho \cdot g \cdot D(y) \cdot A(y)\, dy \\
&= \int_0^{20} 62.5 \cdot (20 - y) \cdot \pi y^2 \cdot \frac{9}{100}\, dy \\
&= 62.5 \cdot \pi \cdot \frac{9}{100} \int_0^{20} (20 - y) \cdot y^2\, dy \\
&= 5.625 \cdot \pi \int_0^{20} 20y^2 - y^3\, dy \\
&= 5.625 \cdot \pi \left(\frac{20y^3}{3} - \frac{y^4}{4} \right) \Big|_0^{20} \\
&= 5.625 \cdot \pi \cdot \frac{40000}{3} \\
&= 75000\pi \\
&\approx 235619 \text{ ft-lbs.}
\end{aligned}
$$

Solutions should show all of your work, not just a single final answer.

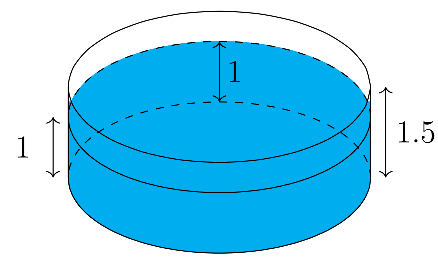
2. A cable that weighs 4 lb/ft is used to lift a 300 lb lump of coal up from the bottom of a mineshaft that is 1000 ft deep. Determine the work needed to bring the coal to the top of the mineshaft using the cable. (Hint: Compute the work done in lifting the cable and the coal separately.)

   (a) Draw and label a picture of the situation.

   (b) Compute the work done in lifting the coal to the top of the mineshaft by itself.

   (c) What is the weight of a Δx slice of the rope?

3. A circular swimming pool with diameter of 8 m. and height of 1.5 m. contains water to a depth of 1 m. Compute the amount of work required to pump all the water out of the pool over the side, giving your final answer in joules to the nearest integer. Consider the density of water to be 1000 kg/m^3 and the acceleration due to gravity to be $g = 9.8\,\text{m}/\text{sec}^2$. Let $y = 0$ at the top of the tank.



(a) What is $A(y)$?

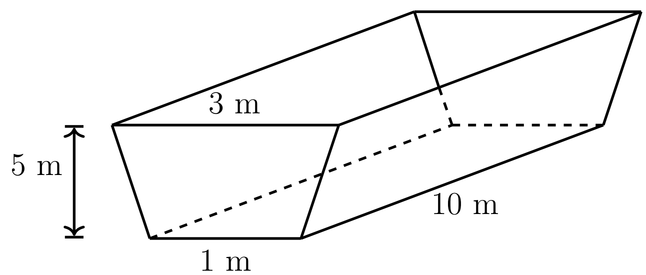
(b) What is $D(y)$?

(c) What is $\rho \cdot g$?

(d) What should the bounds a and b be?

(e) Set up the integral to find work needed to empty the swimming pool.

4. Suppose the following tank is full of oil with a density of 900 kg/m^3. Set up but **do not evaluate** the integral to find the amount work required to pump all the oil out of the tank.



5. T/F (with justification): The work required to stretch a spring having spring constant k a distance x from its equilibrium (rest) position is kx.