11.9 Representations of Functions as Power Series

Power Series, Derivatives, and Integrals. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Alternating Series Estimation Theorem. If $s = \sum(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

(i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Geometric Power Series.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1.$$

1. **Example:** Find a power series centered at $x = 0$ for the following functions and find the interval of convergence of the power series.

$$\frac{1}{2-5x}$$

Thinking about the problem:

Which technique should I use to determine the power series for the function given? Have I seen a problem similar to this one before? If so, which technique did I use?

I see that my function looks very similar to the function $\frac{1}{1-x}$, so I will alter my function to match that form. To make my function look like $\frac{1}{1-x}$, I will factor a 2 out of the denominator so my function looks like $\frac{1}{2(1-5x/2)}$. At this point, I think my function is close enough to $\frac{1}{1-x}$ so that I can find a power series. I note that the interval of convergence of the power series of $\frac{1}{1-x}$ is $(-1, 1)$, so my power series should have a similar interval of convergence.

Doing the problem:

The problem asks to find a power series of a function. $f(x) = \dfrac{1}{2-5x}$ can simply f as follows: I see that

$$\frac{1}{2-5x} = \frac{1}{2(1-5x/2)} = \frac{1}{2} \cdot \frac{1}{1-5x/2}.$$

Using the PS representation $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$, then $\dfrac{1}{1-5x/2} = \displaystyle\sum_{n=0}^{\infty} (5x/2)^n$. So I find that

$$\frac{1}{2-5x} = \frac{1}{2} \cdot \frac{1}{1-5x/2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{5x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(5x)^n}{2^{n+1}}.$$

I have now found a power series centered at $x = 0$ for $\dfrac{1}{2-5x}$. Next, I will find the interval of convergence. Since the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} x^n$ is $|x| < 1$, then the radius of convergence for $\displaystyle\sum_{n=0}^{\infty}\left(\frac{5x}{2}\right)^n$ is $\left|\dfrac{5x}{2}\right| < 1$, so $|x| < \dfrac{2}{5}$. Therefore the interval of convergence of $\displaystyle\sum_{n=0}^{\infty}\left(\frac{5x}{2}\right)^n$ is $\left(-\dfrac{2}{5}, \dfrac{2}{5}\right)$ and so the interval of convergence of $\displaystyle\sum_{n=0}^{\infty}\frac{(5x)^n}{2^{n+1}}$ is also $\left(-\dfrac{2}{5}, \dfrac{2}{5}\right)$.

2. Find a power series centered at $x = 0$ for the following functions and find the interval of convergence of the power series.

$$\frac{1}{1+x^4}$$

Solutions should show all of your work, not just a single final answer.

3. Find a power series centered at $x = 0$ for the following functions and find the interval of convergence of the power series.

$$\frac{1}{(1-x)^3}$$

(a) What is the power series centered at $x = 0$ for $\dfrac{1}{1-x}$? What is the radius of convergence for this power series?

(b) What is the second derivative of $\dfrac{1}{1-x}$?

(c) What is the power series of the second derivative of $\dfrac{1}{1-x}$?

(d) What is the radius of convergence of the power series in (c)?

(e) Use (b) and (c) to find the power series of $\dfrac{1}{(1-x)^3}$.

4. Use power series to estimate $\displaystyle\int_0^{1/2} \frac{dx}{1+x^4}$ to within .00001 by the following steps.

(a) Express $\displaystyle\int \frac{dx}{1+x^4}$ as a power series, starting with the power series you found in 3.

(b) Find the radius of convergence of the power series in part a.

(c) Use the previous parts and the Alternating Series Estimation Theorem to estimate $\displaystyle\int_0^{1/2} \frac{dx}{1+x^4}$ to within .00001. Round your *estimate* to 5 digits.

5. T/F (with justification)

If $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 3 then $\displaystyle\sum_{n=0}^{\infty} c_n x^{2n}$ has radius of convergence 9.