11.8 Power Series

Definition. A Power Series (PS) is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$

where x is a variable and each c_n is a constant called coefficients of the series.

A series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is called a PS centered at "a".

Power Series. For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$

(ii) The series converges for all x

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

1. **Example:** Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=0}^{\infty} 7^{n+1} x^n$$

Thinking about the problem:

Which test should I use to determine the interval of convergence and why? Have I seen a problem similar to this one before? If so, which test did I use?

I think I will use the Ratio Test to determine the interval of convergence and the radius of convergence. I need to remember that the Ratio Test does not tell me if the power series converges on the endpoints of the interval I find. So, I will need to test whether the series $\sum_{n=0}^{\infty} 7^{n+1}x^n$ converges or diverges at the endpoints of the interval.

Doing the problem:

The problem asks for an interval of convergence of a power series. I will start by applying the Ratio Test. I know that $a_n = 7^{n+1}x^n$, so $a_{n+1} = 7^{n+2}x^{n+1}$. Therefore,

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{7^{n+2}x^{n+1}}{7^{n+1}x^n}\right| \\ &= \lim_{n\to\infty}|7x| \\ &= |7x|.\end{aligned}$$

By the Ratio Test, I know that the series converges when $|7x| < 1$. The inequality $|7x| < 1$ is true for $(-1/7, 1/7)$, but I still need to test the end points to see if the power series converges when $x = -1/7$ or $x = 1/7$. First, I let $x = 1/7$. Then the series becomes

$$\sum_{n=0}^{\infty} 7^{n+1}\left(\frac{1}{7}\right)^n = \sum_{n=0}^{\infty} 7 \cdot 7^n \left(\frac{1}{7}\right)^n = \sum_{n=0}^{\infty} 7.$$

By the Test for Divergence, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 7 \neq 0$, so I can conclude that $\sum\limits_{n=0}^{\infty} 7^{n+1}x^n$ diverges when $x = 1/7$. Next, I let $x = -1/7$. Then the series becomes

$$\sum_{n=0}^{\infty} 7^{n+1}\left(-\frac{1}{7}\right)^n = \sum_{n=0}^{\infty} 7 \cdot 7^n \left(-\frac{1}{7}\right)^n = \sum_{n=0}^{\infty} 7 \cdot (-1)^n.$$

By the Test for Divergence, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 7 \cdot (-1)^n \neq 0$, so I can conclude that $\sum\limits_{n=0}^{\infty} 7^{n+1}x^n$ diverges when $x = -1/7$. Therefore the interval of convergence of $\sum\limits_{n=0}^{\infty} 7^{n+1}x^n$ is $(-1/7, 1/7)$.

Solutions should show all of your work, not just a single final answer.

2. For the following power series:

$$\sum_{n=0}^{\infty} 7^{n+1} x^{2n}$$

(a) What is a_n in this series?

(b) Evaluate the limit $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

(c) For what values of x is $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$?

(d) Does this mean that we can conclude anything about the convergence of the power series on the endpoints of the interval in (d)?

3. For the following power series:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$$

(a) For what values of x is $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$?

(b) Write out the two series obtained by substituting each endpoint from the interval obtained in part (a) into the PS $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$.

(c) Which test or tests could you use to determine whether the series in (b) are convergent or divergent?

(d) Determine whether the series found in (b) are convergent or divergent (*Hint:* You should be determining the convergence or divergence of two different series).

4. Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=0}^{\infty} \frac{x^n}{(2n+1)!}$$

5. T/F (with justification)

If $\sum_{n=0}^{\infty} c_n$ converges then $\sum_{n=0}^{\infty} c_n x^n$ converges when $|x| < 1$.