Math 1132 Worksheet 11.4 Name: ________________________ Discussion Section: _____

11.4 The Comparison Tests

Comparison Theorem. For series $\sum a_n$ and $\sum b_n$ with positive terms

(a) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n then $\sum a_n$ is also convergent.

(b) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n then $\sum a_n$ is also divergent.

Limit Comparison Test. For series $\sum a_n$ and $\sum b_n$ with positive terms. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$ where $L > 0$ is a finite number then either both series converge or both series diverge.

1. **Example:** Determine whether the series $\sum\limits_{n=2}^{\infty} \frac{n^3}{n^4+1}$ converges or diverges.

   Thinking about the problem:

   Which test should I use to determine whether the series converges or diverges and why?

   Have I seen a problem similar to this one before? If so, which test did I use?

   To determine which test I should use I will focus on the nth term, that is, $a_n = \frac{n^3}{n^4+1}$.

   In this case I think I can use either the Integral Test or one of the Comparison Tests. I will try a Comparison Test. I will need a test series $\sum b_n$ to compare to $\sum a_n = \sum \frac{n^3}{n^4+1}$ to. So, what should $\sum b_n$ be? To define b_n I look at the leading terms of $\sum a_n = \sum \frac{n^3}{n^4+1}$ and take $b_n = \frac{n^3}{n^4} = \frac{1}{n}$ giving me the test series $\sum b_n = \sum \frac{1}{n}$, which is a divergent p-series.

Next, I apply the test. I will start with Limit Comparison Test because often the inequalities required in the other comparison test do not come easily. (For instance, $a_n = \dfrac{n^3}{n^4+1} \geq \dfrac{1}{n} = b_n$ does not hold.) I note that the terms of the two series are positive terms and then compute $L = \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\frac{n^3}{n^4+1}}{\frac{1}{n}}$.

Doing the problem:

The problem asks whether the series converges or diverges. I note that the terms of the two series are positive terms and choose to apply the Limit Comparison Test (LCT). I know that $a_n = \dfrac{n^3}{n^4+1}$ and I define $b_n = \dfrac{n^3}{n^4} = \dfrac{1}{n}$. My test series $\sum b_n = \sum \dfrac{1}{n}$ is a divergent p-series. $L = \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\frac{n^3}{n^4+1}}{\frac{1}{n}} = \lim\limits_{n\to\infty} \left(\dfrac{n^3}{n^4+1}\right)\left(\dfrac{n}{1}\right) = \lim\limits_{n\to\infty} \dfrac{n^4}{n^4+1} = \lim\limits_{n\to\infty} \dfrac{1}{1+1/n^4} = 1$. By the LCT, since $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1 > 0$, then either both series converge or both series diverge. Since the test series (i.e. $\sum \dfrac{1}{n}$) diverges, then $\sum a_n = \sum \dfrac{n^3}{n^4+1}$ also diverges.

2. Determine whether the series $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ converges or diverges.

(a) Which test or tests could you use to determine whether the series converges or diverges?

(b) State the test you would use to decide whether the series converges or diverges.

(c) Determine whether the series $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ converges or diverges.

3. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{3n^4+1}}$ converges or diverges.

(a) Which test would you use to determine whether the series converges or diverges?

(b) Decide whether the series converges or diverges.

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{n+2}{(n+1)^3}$ converges or diverges.

5. T/F (with justification) If $0 < a_n \leq b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.