11.3 The Integral Test and Estimates of Sums

The Integral Test. Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Remainder Estimate for the Integral Test. Suppose $f(k) = a_k$, where f is a continuous, positive, and decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

1. **Example:** Determine whether or not the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

Thinking about the problem:

Which test should I use to determine whether the series converges or diverges and why?

Have I seen a problem similar to this one before? If so, which test did I use?

To determine which test to use I will focus on the nth term, that is, $a_n = \frac{n}{n^2+1}$. In this case, I think I can use the Integral Test because I can integrate $f(x) = \frac{x}{x^2+1}$ by substitution. I need to check the conditions of the Integral Test to see if I can apply it. This involves checking that $f(x) = \dfrac{x}{x^2+1}$ is positive, continuous, and decreasing for $x \geq 1$. I need to show my work in showing that $f(x)$ satisfies the conditions, then I will be able to take the improper integral $\displaystyle\int_1^\infty f(x)\,dx$ to see if my series converges or diverges.

Doing the problem:

The problem asks whether the series converges or diverges. I note that the function $f(x) = \dfrac{x}{x^2+1}$ is positive for $x \geq 1$ since both x and x^2+1 are positive for $x \geq 1$. I also see that $f(x)$ is continuous for $x \geq 1$ since there is no point where I cannot find a value for $f(x)$ in this interval. Finally, I need to check if $f(x)$ is decreasing. I can do this by taking the derivative $f'(x)$ and showing that it is negative. I see that

$$f'(x) = \frac{x^2+1-(2x)x}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}.$$

Since $(x^2+1)^2$ is always positive, $f'(x)$ is negative when $1-x^2$ is negative.

$$1 - x^2 < 0$$

$$1 < x^2$$

$$1 < x.$$

So I see that $f(x)$ is decreasing for $x > 1$. Thus, since $f(x)$ is decreasing for $x > 1$, all 3 conditions are satisfied and we can apply the Integral Test.

Now that I have checked the conditions of the Integral Test, I can apply the test to my series. To determine if $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converges or diverges, I need to evaluate improper integral $\int_1^{\infty} \frac{x}{x^2+1}\, dx$. So,

$$\int_1^{\infty} \frac{x}{x^2+1}\, dx = \lim_{a\to\infty} \int_1^{a} \frac{x}{x^2+1}\, dx$$

Let $u = x^2 + 1$ so $du = 2x\, dx$ to find

$$\begin{aligned}
&= \lim_{a\to\infty} \int_2^{a^2+1} \frac{x}{u} \cdot \frac{du}{2x} \\
&= \frac{1}{2} \lim_{a\to\infty} \int_2^{a^2+1} \frac{du}{u} \\
&= \frac{1}{2} \lim_{a\to\infty} (\ln |u|) \Big|_2^{a^2+1} \\
&= \frac{1}{2} \left(\lim_{a\to\infty} \ln |a^2+1| - \ln |2| \right) \\
&= \infty.
\end{aligned}$$

Therefore, since $\int_1^{\infty} \frac{x}{x^2+1}\, dx$ is divergent, $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is divergent.

Solutions should show all of your work, not just a single final answer.

2. Determine whether or not the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$$

(a) Which test or tests could you use to determine whether the series converges or diverges?

(b) State the test you would use to decide whether the series converges or diverges.

(c) List the conditions of the test you would use in part (b).

(d) Check that the conditions hold for the $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$.

3. Determine whether or not the following series converges or diverges. (*Hint:* Check any and all conditions necessary before applying a test to the series.)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$$

4. Consider the series $\sum_{n=1}^{\infty} \frac{n}{3^n}$

(a) Verify that the integral test can be used to decide if this series converges and then use the integral test to show it converges. (*Hint:* Use integration by parts to evaluate the integral.)

(b) Determine an explicit upper bound for the remainder when estimating the series by the 100th partial sum, i.e. $N = 100$.

(c) Determine an explicit upper bound for the remainder when estimating the series by the Nth partial sum, in terms of N.

(d) Find an N for which the Nth partial sum has a remainder that is less than .001, and then compute that partial sum to 4 digits.

5. T/F (with justification): If $f(x)$ is continuous, positive, and decreasing for $x \geq 1$, and $\int_1^\infty f(x)\,dx$ converges then $\sum_{n=1}^{\infty} f(n) = \int_1^\infty f(x)\,dx$.