11.1 Sequences

Remember: A sequence $\{a_n\}$ has a limit L and we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$ if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say that the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

1. **Example**: Determine whether the following sequence is convergent or divergent.

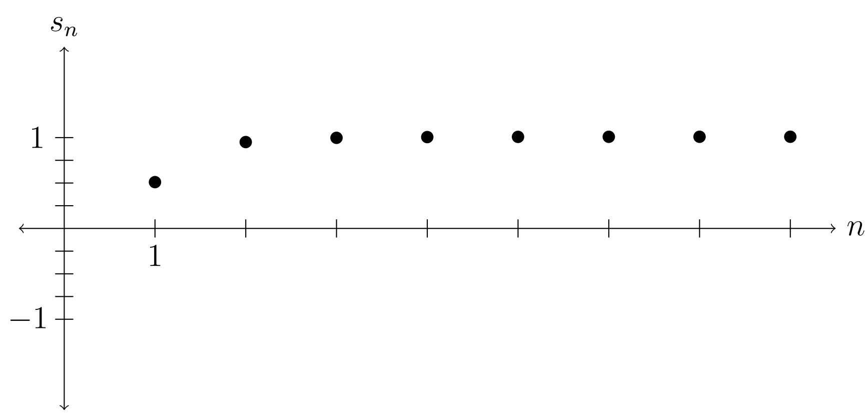
$$\left\{\frac{n^4}{n^4+1}\right\} \text{ for } n \geq 1.$$

Thinking about the problem:

How should I determine whether the sequence converges or diverges and why? Have I seen a problem similar to this one before? If so, what did I do?

First, I want to see if $\{a_n\}$ converges or diverges. So I will plot some of the a_n to get a sense of what $\{a_n\}$ looks like.

n	1	2	3	4	5	6	7	8
a_n	.5	.94118	.9878	.99611	.9984	.99923	.99958	.99976



Next, I will take the limit of $\{a_n\}$ as $n \to \infty$ to see if it exists. I could use the graph to help verify whether the sequence converges or diverges.

Doing the problem:

I will take the limit of $\left\{ \dfrac{n^4}{n^4+1} \right\}$ to see if the sequence converges or diverges. I see that

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^4}{n^4+1} &= \lim_{n\to\infty} \frac{n^4}{n^4+1} \cdot \frac{1/n^4}{1/n^4} \\
&= \lim_{n\to\infty} \frac{1}{1+1/n^4} \\
&= 1.
\end{aligned}$$

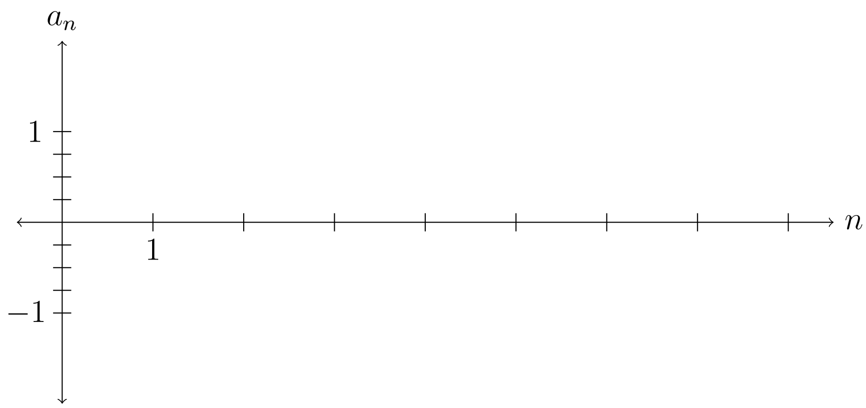
Since the limit exists, this shows that $\left\{ \dfrac{n^4}{n^4+1} \right\}$ converges.

Solutions should show all of your work, not just a single final answer.

2. Let $a_n = \dfrac{(-1)^{n-1}}{n}$.

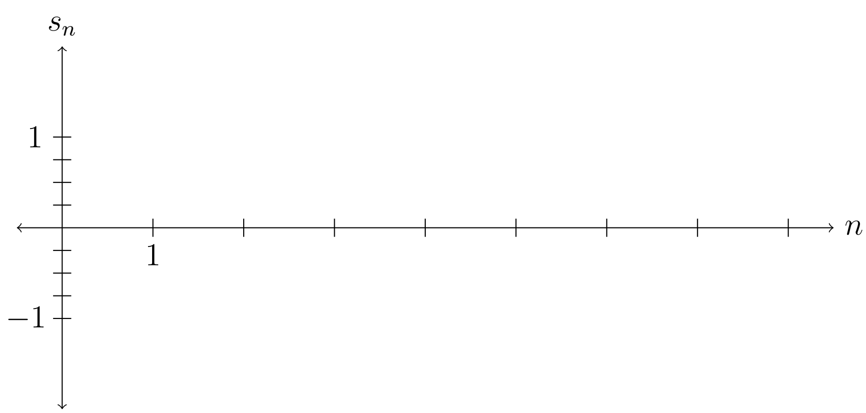
(a) Plot a_n vs. n for $n = 1, 2, 3, 4, 5, 6, 7, 8$.

n	1	2	3	4	5	6	7	8
a_n								



(b) Compute $s_n = \sum_{k=1}^{n} a_k$ for $n = 1, 2, \ldots, 8$ to three digits after the decimal point and then plot s_n vs. n for $n = 1, 2, \ldots, 8$.

n	1	2	3	4	5	6	7	8
s_n								



3. Determine the limit of the sequence or state the limit does not exist. If there is a limit, show the calculations that explain how you are finding the limit.

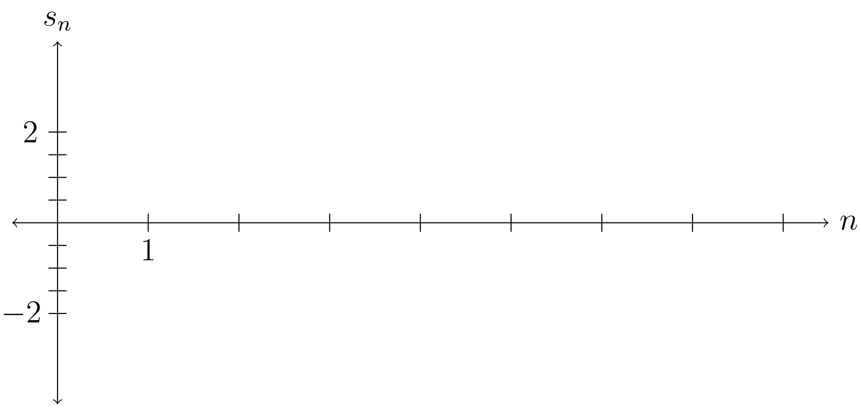
$$\left\{ \left(1 + \frac{1}{2n}\right)^n \right\} \text{ for } n \geq 1.$$

Note:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

(a) Plot a_n vs. n for $n = 1, 2, 3, \ldots, 8$.

n	1	2	3	4	5	6	7	8
a_n								



(b) By looking at the plot, does $\{a_n\}$ look like it converges? Does it look like it diverges? If it looks like it converges, what do you think it converges to?

(c) Calculate $\lim_{n\to\infty} a_n$ if it exists.

(*Hint:* $\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^n = L$ implies that $\lim_{n\to\infty} \left(n \cdot \ln\left(1 + \frac{1}{2n}\right)\right) = \ln L$).

4. Determine the limit of the sequence or state the limit does not exist. If there is a limit, show the calculations that explain how you are finding the limit.

$$\left\{ \frac{\cos n}{\sqrt{n}} \right\} \text{ for } n \geq 1.$$

5. T/F (with justification): Every bounded sequence is convergent.