Math 1132 Worksheet 10.4 Name: ________________________ Discussion Section: _____

Areas in Polar Coordinates

Area. The formula for the area A of a polar region $\mathcal{R}$ is

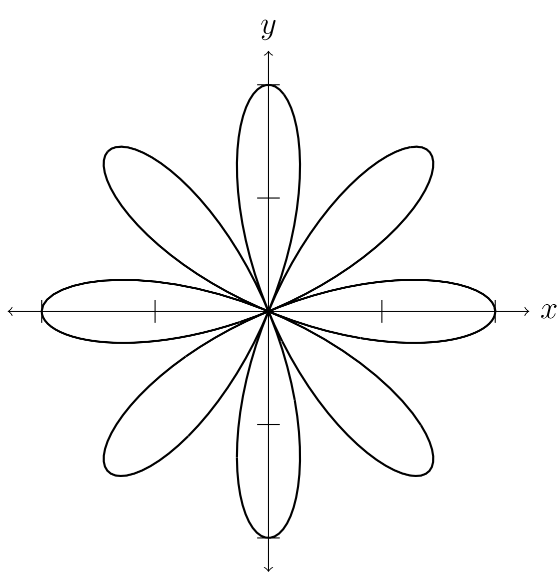
$$A = \int_a^b \frac{1}{2}[f(\theta)]^2\, d\theta = \int_a^b \frac{1}{2} r^2\, d\theta.$$

Caution: The fact that a single point has many representations in polar coordinates sometimes makes it difficult to find all the points of intersection of two polar curves. Thus, to find *all* points of intersection of two polar curves, it is recommended that you draw the graphs of both curves.

Note: The area of the region bounded by two polar equations $r = f(\theta)$, $r = g(\theta)$, $\theta = a$ and $\theta = b$ is given by

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2\, d\theta - \int_a^b \frac{1}{2}[g(\theta)]^2\, d\theta.$$

1. **Example:** Below is a graph of $r = 2\cos 4\theta$ for $0 \le \theta \le 2\pi$. Markings at ± 1 and ± 2 are on the x and y axes.



Determine the area of the graph.

Thinking about the problem:

How should I approach this problem? Have I seen a problem like this before? If so, how did I approach it?

I know the formula for the area of a polar region is

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2\, d\theta = \int_a^b \frac{1}{2}r^2\, d\theta.$$

I also know that $0 \le \theta \le 2\pi$ and $r = 2\cos 4\theta$, so my area should be

$$A = \int_0^{2\pi} \frac{1}{2}[2\cos 4\theta]^2\, d\theta.$$

Note that the graph is symmetric, so the area of the half petal from $0 \le \theta \le \frac{\pi}{8}$ should be $\frac{1}{16}$th of the entire area. I can find both areas independently to check that the area enclosed by $r = 2\cos 4\theta$ for $0 \le \theta \le 2\pi$ is the same as the region on the graph.

Doing the problem:

The problem asks to find the area of a polar region given by $r = 2\cos 4\theta$ for $0 \le \theta \le 2\pi$. I apply the formula

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2\, d\theta = \int_a^b \frac{1}{2}r^2\, d\theta$$

to find

$$\begin{aligned}
A &= \int_a^b \frac{1}{2}r^2\, d\theta \\
&= \int_0^{2\pi} \frac{1}{2}[2\cos 4\theta]^2\, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 4\cos^2(4\theta)\, d\theta
\end{aligned}$$

$$= \frac{1}{2}\int_0^{2\pi} 4 \cdot \frac{1}{2}(1+\cos(8\theta))\,d\theta$$

$$= \int_0^{2\pi} 1+\cos(8\theta)\,d\theta$$

$$= \left(\theta + \frac{1}{8}\sin(8\theta)\right)\Big|_0^{2\pi}$$

$$= \left(2\pi + \frac{1}{8}\sin(8\cdot 2\pi)\right) - \left(0 + \frac{1}{8}\sin(8\cdot 0)\right)$$

$$= 2\pi$$

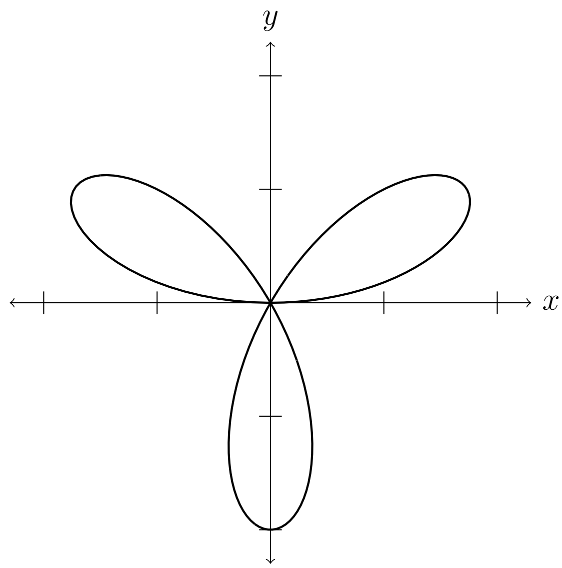
So the area enclosed by $r = 2\cos(4\theta)$ is 2π. Note that

$$A = 16 \cdot \int_0^{\pi/8} \frac{1}{2}[2\cos 4\theta]^2\,d\theta = 2\pi$$

So I can conclude that the area enclosed by $r = 2\cos 4\theta$ for $0 \le \theta \le 2\pi$ is indeed the same as the region on the graph.

Solutions should show all of your work, not just a single final answer.

2. Below is the graph of $r = 2\sin(3\theta)$ (a 3-leaf rose). Markings at ± 1 and ± 2 are on the x and y axes.

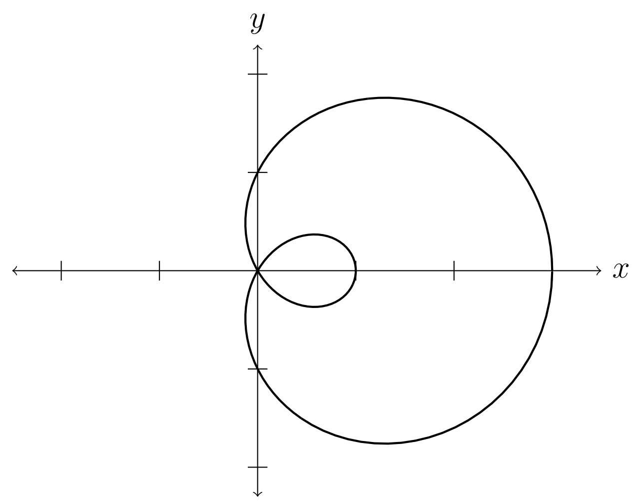


(a) Mark points on the rose where $\theta = 0,\ \pi/6,\ \pi/4,\ \pi/2$, and $3\pi/4$, and draw arrows on each leaf to indicate the direction in which it is traced out as θ increases.

(b) Determine the smallest positive angle θ at which the rose passes through the origin, and use this to help you set up an integral for the area of the leaf in the first quadrant.

(c) Compute the integral in part c. (Hint: Recall from Section 7.2 how to integrate $\sin^2 u$ by writing it in terms of $\cos(2u)$.)

3. Below is the graph $r = 1 + 2\cos\theta$. Unit markings are on the x and y axes (Note: This curve is called a limaçon).



Set up, **but do not evaluate**, an integral in terms of θ for the area enclosed by the inner loop of the limaçon. Be sure the bounds of integration are correct.

4. Set up, but do not evaluate, an integral in terms of θ for the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

5. T/F (with justification)

   If the polar curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$ completely encloses a region, the area of the region is $\displaystyle\int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 \, d\theta$.