10.3 Polar Coordinates

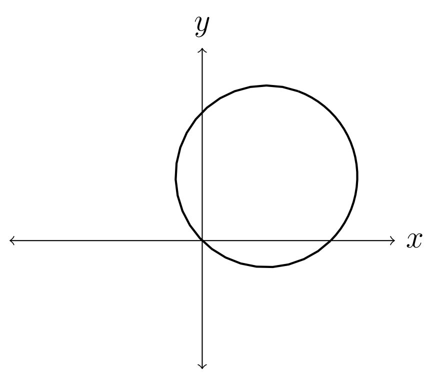
Polar Coordinates/Cartesian Coordinates. To find Cartesian coordinates (x, y) of a point when its polar coordinates (r, θ) are known, use $x = r\cos\theta, y = r\sin\theta$.

To find polar coordinates (r, θ) of a point when its Cartesian coordinates (x, y) are known, use $r^2 = x^2 + y^2, \tan\theta = \dfrac{y}{x}$.

Derivatives.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Example: Below is the plot of the polar equation $r = \sin\theta + \cos\theta$.



Fill in the table below, use it to determine the orientation of the curve (direction of increasing θ), and find the equation of the tangent line to the curve at $(x, y) = (0, 0)$.

θ	$\sin\theta + \cos\theta$	(r, θ)	(x, y)
0			
$\pi/4$			
$\pi/2$			
π			

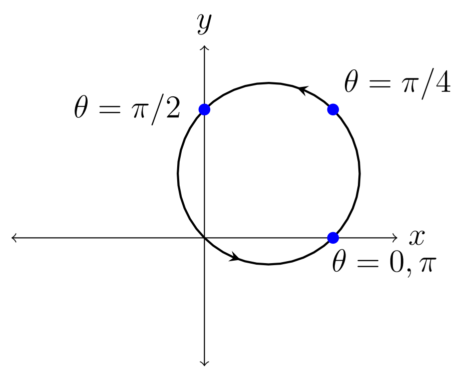
Thinking about the problem:

We can fill in the table using the polar equation and the conversion formulas from polar to Cartesian coordinates. To find the tangent line at $(x, y) = (0, 0)$ we will find the value of θ at which the curve passes through the origin and then compute dy/dx at that value of θ.

Doing the problem:

Using the formulas $x = r\cos\theta$ and $y = \sin\theta$ we find the following:

θ	$\sin\theta + \cos\theta$	(r, θ)	(x, y)
0	1	$(1, 0)$	$(1, 0)$
$\pi/4$	$\sqrt{2}$	$(\sqrt{2}, \pi/4)$	$(1, 1)$
$\pi/2$	1	$(1, \pi/2)$	$(0, 1)$
π	-1	$(-1, \pi)$	$(1, 0)$



From the marked points already put on the circle, it is natural to guess $\theta = 3\pi/4$ will correspond to the origin, and indeed for this value of θ we have $\sin\theta = 1/\sqrt{2}$ and $\cos\theta = -1/\sqrt{2}$, so $r = 1/\sqrt{2} - 1/\sqrt{2} = 0$ and thus the point is the origin.

To find the equation of the tangent line, we compute $\dfrac{dy}{d\theta}$ and $\dfrac{dx}{d\theta}$ at $\theta = 3\pi/4$ and then take the ratio to get $\dfrac{dy}{dx}$. First, since $y = r\sin\theta$ and $r = \sin\theta + \cos\theta$,

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = (\cos\theta - \sin\theta)\sin\theta + r\cos\theta = \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 0 = -1$$

and since $x = r\cos\theta$ and $r = \sin\theta + \cos\theta$,

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta + r(-\sin\theta) = (\cos\theta - \sin\theta)\cos\theta - r\sin\theta = \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) + 0 = 1.$$

Therefore $\left.\frac{dy}{dx}\right|_{\theta=3\pi/4} = \frac{-1}{1} = -1.$

Therefore the tangent line through the origin has slope -1, so its equation is $y = -x$.

Solutions should show all of your work, not just a single final answer.

1. Here are three points in polar coordinates:

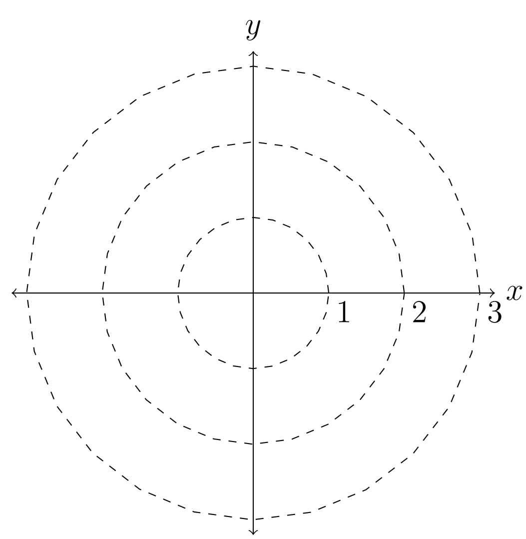
   (a) $P = (2, \pi/4)$, (b) $Q = (-3, 3\pi/4)$, (c) $R = (-1, -\pi/3)$.

   For each of these do the following.

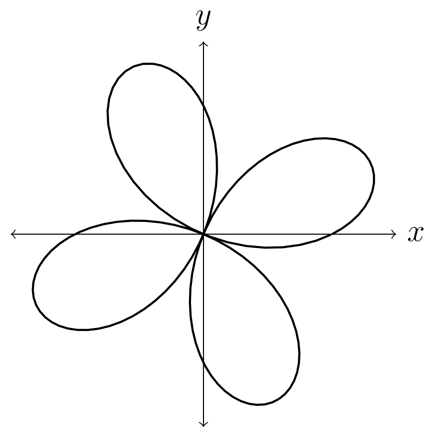
   (i) Convert it to Cartesian coordinates (give exact values, not approximations).

   (ii) Plot the point on the axes below.

   (iii) Give two additional representations of the point in polar coordinates.



2. Here is a plot of $r = \sin(2\theta) + \cos(2\theta)$.

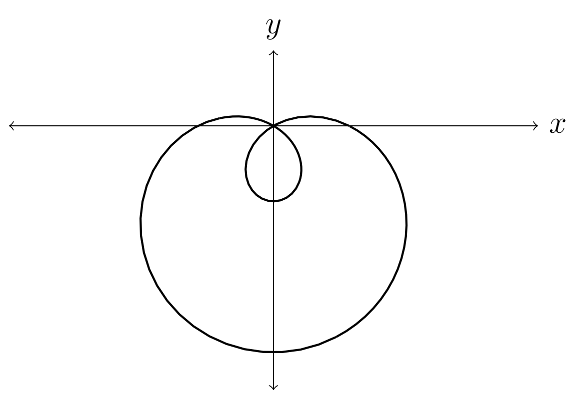


Fill in the table below.

θ	$\sin(2\theta) + \cos(2\theta)$	(r, θ)	(x, y)
0			
$\pi/2$			
π			
$3\pi/2$			
2π			

Based on this table, and additional data if it seems needed, draw arrows on the curve (including at least one on each loop) to indicate the direction of increasing θ.

3. Here is a plot of the polar equation $r = 1 - 2\sin\theta$.



(a) Mark points on the curve for $\theta = 0, \pi/4, \pi/2$, and π, and then draw arrows on the curve (including on the big and small loops) to indicate the direction of increasing θ.

(b) Find the equation of the tangent line to this curve at the point $(x, y) = (1, 0)$.

4. T/F (with justification)

   Each point in the plane besides the origin can be written in polar coordinates (r, θ) with $r < 0$.