Math 1132 - Calculus 2 : Summary of convergence tests for infinite series

This is a guide for determining convergence or divergence of a series. You must be able to use these during exams.

Series or Test	Form of the Series	Condition implying Convergence	Condition implying Divergence	Comments
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\|r\| < 1$	$\|r\| \geq 1$	If $\|r\| < 1$, then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$
Test for Divergence	$\sum_{n=1}^{\infty} a_n$	Does not apply	$\lim_{n\to\infty} a_n \neq 0$	Cannot be used to prove convergence
Integral Test	$\sum_{n=1}^{\infty} a_n$ with $a_n = f(n)$ and $f(x)$ for $x \geq 1$ is continuous, positive, and decreasing	$\int_1^{\infty} f(x)\,dx < \infty$	$\int_1^{\infty} f(x)\,dx = \infty$	The value of the integral is not the value of the series.
p-series ($p > 0$)	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	This is proved by the integral test. It is useful for comparison tests.
Comparison Test	$\sum_{n=1}^{\infty} a_n$, where $a_n > 0$ and $b_n > 0$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	$\sum_{n=1}^{\infty} a_n$ is given; you supply $\sum_{n=1}^{\infty} b_n$
Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$, where $a_n > 0$ and $b_n > 0$	$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, $\sum_{n=1}^{\infty} b_n$ converges.	$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, $\sum_{n=1}^{\infty} b_n$ diverges.	$\sum_{n=1}^{\infty} a_n$ is given; you supply $\sum_{n=1}^{\infty} b_n$
Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^n b_n$ or $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, where $b_n > 0$	$b_{n+1} \leq b_n$, $\lim_{n\to\infty} b_n = 0$	$\lim_{n\to\infty} b_n \neq 0$	
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| < 1$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| > 1$	Inconclusive if $\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| = 1$
Absolute Convergence	$\sum_{n=1}^{\infty} a_n$, where many $a_n < 0$ and Alternating Test useless	$\sum_{n=1}^{\infty} \|a_n\|$ converges	$\lim_{n\to\infty} \|a_n\| \neq 0$	Use other tests on $\sum_{n=1}^{\infty} \|a_n\|$

Two of these tests provide **error estimates** on $|s - s_n|$ that you should know:

1. In the integral test, $|s - s_n| \leq \int_n^{\infty} f(x)\,dx$, so if you want an n making $|s - s_n|$ small, find an n making $\int_n^{\infty} f(x)\,dx$ small.
2. When an alternating series passes the alternating series test, $|s - s_n| \leq b_{n+1}$: the error is bounded by the **first** missing term. So if you want an n making $|s - s_n|$ small, find an n making b_{n+1} small.

Convergence of a series is a property of its "tail": any convergence test can be used if initial terms fail the hypotheses:

1. For integral test, if $f(x)$ is decreasing only for $x \geq 3$, then work with $\int_3^{\infty} f(x)\,dx$.
2. Comparison test works "as is" if $a_n \leq b_n$ or $b_n \leq a_n$ only for large n, such as $n \geq 3$.
3. Alternating Series test works "as is" even if $b_{n+1} \leq b_n$ only for large n, such as $n \geq 3$.