Approximate Integration

Solutions should show all of your work, not just a single final answer.

Example. (a) Apply the Trapezoidal Rule to $\int_1^3 \sqrt{x}\,dx$ using $n = 4$ subintervals, rounding your approximation to 5 digits after the decimal point.

(b) Use the error bound for the Trapezoidal Rule to determine an n such that the Trapezoidal Rule is *guaranteed* by the error bound to be with .01 of the value of the integral.

Solution.

(a) Using $f(x) = \sqrt{x}$, the Trapezoidal Rule with $n = 4$ is

$$\begin{aligned}\frac{b-a}{2n}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)) &= \frac{2}{8}(f(1) + 2f(1.5) + 2f(2) + \\ &\qquad 2f(2.5) + f(3)) \\ &\approx 2.79306.\end{aligned}$$

(b) An upper bound on the error from the Trapezoidal Rule with n intervals is $\frac{K(b-a)}{12}(\Delta x)^2$, where $\Delta x = (b-a)/n$ and K is an upper bound on $|f''(x)|$ for all x in $[a, b]$. In our problem, $f(x) = \sqrt{x}$, so $f''(x) = -\frac{1}{4}x^{-3/2}$. For $1 \le x \le 3$, we have $x^{-3/2} \le 1$, so $|f''(x)| \le 1/4$ when $1 \le x \le 3$. Thus we can use $K = 1/4$, so the trapezoidal error bound is $\frac{(1/4)(3-1)}{12}(\frac{2}{n})^2 = \frac{1}{6n^2}$. Having the error be less than .01 means

$$\frac{1}{6n^2} < .01 \iff n^2 > \frac{1}{6(.01)} \iff n > \sqrt{\frac{1}{.06}} \approx 4.082,$$

so for $n = 5$ or more the trapezoidal approximation will be within .01 of the integral.

1. (a) Apply the Trapezoidal Rule to $\int_2^3 x \sin x\,dx$ using $n = 4$ subintervals, rounding your approximation to 5 digits after the decimal point. (Use radian mode in a calculator!)

   (b) Use the error bound for the Trapezoidal Rule to determine an n such that the Trapezoidal Rule is *guaranteed* by the error bound to be with .01 of the value of the integral.

2. (a) Apply Simpson's Rule to $\int_1^2 \sqrt{x}\,dx$ using $n = 4$ subintervals, rounding your approximation to 5 digits after the decimal point.

   (b) Use the error bound for Simpson's Rule to determine an n such that Simpson's Rule is *guaranteed* by the error bound to be with 10^{-6} of the value of the integral. (Remember n must be even.)

3. T/F (with justification) The Trapezoidal Rule for $\int_a^b f(x)\,dx$ has no error if $f(x)$ is linear.