Trigonometric Integrals

Solutions should show all of your work, not just a single final answer.

1. (a) Use the identity $\sin^2 x + \cos^2 x = 1$ and double-angle formula $\cos(2x) = \cos^2 x - \sin^2 x$ to derive formulas for both $\sin^2 x$ and $\cos^2 x$ in terms of $\cos(2x)$.

   (b) Show the double-angle formulas $\sin(2x) = 2 \sin x \cos x$ and $\cos(2x) = \cos^2 x - \sin^2 x$ can each be derived from the other one using differentiation and algebra: differentiate each identity and then use algebra to turn the first formula into the second and to turn the second formula into the first. This shows that if you only remember one of these formulas, you can recover the other one with calculus.

2. Use 1a to work out $\int \sin^2 x \, dx$ and $\int \cos^2 x \, dx$.

3. Use integration by substitution to compute $\int \cos x \sin^2 x \, dx$.

4. Compute $\int \sin^3(5x) \, dx$ using the identity $\sin^2(5x) + \cos^2(5x) = 1$.

5. Compute $\int \sin^2 x \cos^2 x \, dx$ by using the identity $\sin(2x) = 2 \sin x \cos x$.

6. Compute $\int_0^\pi \sin^3 x \, dx$.

7. Show $\int_{-\pi}^{\pi} \sin^2(mx) \, dx = \pi$ and $\int_{-\pi}^{\pi} \cos^2(mx) \, dx = \pi$ for every positive integer m. These are *fundamental* in the mathematical analysis of vibrations. (Hint: Try this first for specific m, such as $m = 3$, but the solution should treat the general case.)

8. T/F (with justification): The value of $\int_{-\pi}^{\pi} \sin^9 x \, dx$ is 0.