University of Connecticut
Department of Mathematics

Math 1132 Practice Exam 2 SOLUTIONS Spring 2016

Name: ____________________

Instructor Name: ____________________ Section: ________

TA Name: ____________________ Discussion Section: ________

Read This First!

- Please read each question carefully. Show **ALL** work clearly in the space provided. In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- Answers must be clearly labeled in the spaces provided after each question. Please cross out or fully erase any work that you do not want graded. The point value of each question is indicated after its statement. No books or other references are permitted.
- Calculators are allowed but you must show all your work in order to receive credit on the problem.

Grading - For Administrative Use Only

Page:	1	2	3	4	5	6	7	8	9	11	12	Total
Points:	6	6	9	8	9	11	12	12	12	6	4	95
Score:												

1. If the statement is always true, circle the printed capital T. If the statement is sometimes false, circle the printed capital F. In each case, write a careful and clear justification or a counterexample.

(a) The geometric series $\sum_{n=4}^{\infty} \left(\frac{1}{3}\right)^n$ converges to $\frac{3}{2}$. (a) F [2]

Solution 1:

If $|r| < 1$ then $\sum_{n=4}^{\infty} r^n = r^4 + r^5 + r^6 + \ldots = r^4(1 + r + r^2 + \ldots) = \frac{r^4}{1-r}$.

Use $r = \frac{1}{3}$: $\sum_{n=4}^{\infty} \left(\frac{1}{3}\right)^n = \frac{(1/3)^4}{1-1/3} = \frac{(1/3)^4}{2/3} = \frac{3}{2(3^4)} = \frac{1}{2(3^3)} = \frac{1}{54}$.

Solution 2: $s_n = \left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^5 + \ldots + \left(\frac{1}{3}\right)^n$ and

$$\left(\frac{1}{3}\right) s_n = \left(\frac{1}{3}\right)^5 + \ldots + \left(\frac{1}{3}\right)^n + \left(\frac{1}{3}\right)^{n+1}.$$

Subtract: $\left(\frac{2}{3}\right) s_n = \left(\frac{1}{3}\right)^4 - \left(\frac{1}{3}\right)^{n+1} \implies s_n = \frac{3}{2}\left(\frac{1}{3^4}\right) - \frac{3}{2}\left(\frac{1}{3}\right)^{n+1}$

Let $n \to \infty$: $\lim_{n\to\infty} s_n = \frac{3}{2}\frac{1}{3^4} - \frac{3}{2}(0) = \frac{1}{2 \cdot 3^3} = \frac{1}{54}$

(b) If $\lim_{n\to\infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} a_n$ converges. (b) F [2]

A counterexample: the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $\lim_{n\to\infty} \frac{1}{n} = 0$

(c) The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$ converges conditionally. (c) F [2]

$\sum_{k=1}^{\infty} \left|\frac{(-1)^k}{k^3}\right| = \sum_{k=1}^{\infty} \frac{1}{k^3}$ converges (p-series for $p = 3 > 1$)

The series is absolutely convergent, not conditionally convergent.

(d) If $\sum_{n=1}^{\infty} |a_n|$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges. (d) F [2]

A counterexample: $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series) but

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges (alternating harmonic series).

What is true:

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges (absolute convergence test).

If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} |a_n|$ diverges.

(e) The fifth degree Taylor polynomial for $\sin(x)$ centered at $a = 0$ [2]

is $x + \frac{x^3}{3!} + \frac{x^5}{5!}$. (e) F

Sign wrong on middle term: $T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

More generally, $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots$

(f) If the power series $\sum_{k=0}^{\infty} a_k \,(x-4)^k$ has a radius of convergence [2]

equal to 2 then $\sum_{k=0}^{\infty} a_k$ diverges. (f) T

The center is at 4 with $R = 2$, so the interval of convergence is one of the following: $(2,6)$, $[2,6]$, $[2,6)$, $(2,6]$.

The series $\sum_{k=0}^{\infty} a_k$ is $\sum_{k=0}^{\infty} a_k(x-4)^k$ when $x = 5$, and 5

is inside all these intervals, so we have convergence of $\sum_{k=0}^{\infty} a_k$.

2. For each multiple choice question, circle the correct answer. There is only one correct choice for each answer. No justification is required.

(a) Which of the following sequences is both bounded and monotonic? [3]

(i) $a_n = n^2$

(ii) $a_n = \dfrac{n}{n+1}$

(iii) $a_n = \dfrac{\sin(\pi n)}{n}$

(iv) $a_n = \dfrac{n}{\sqrt{n+1}}$

(v) None of the above

(i) $a_n = n^2$ is not bounded: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty$

(ii) $a_n = \dfrac{n}{n+1}$ is bounded and monotonic (this is the answer).

Bounded: $0 < \dfrac{n}{n+1} < \dfrac{n+1}{n+1} = 1$

Monotonic:

$f(x) = \dfrac{x}{x+1} \Longrightarrow f'(x) = \dfrac{x+1-x}{(x+1)^2} = \dfrac{1}{(x+1)^2} > 0 \Longrightarrow f(x)$ increasing

so $a_n = f(n) = \dfrac{n}{n+1}$ is increasing

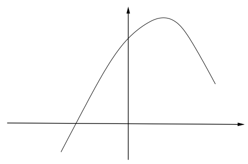
(iii) $a_n = \dfrac{\sin(\pi n)}{n}$ is not monotonic: it is 0 for all n, so it's constant. A monotonic sequence is increasing or decreasing. (The sequence is bounded.)

(iv) $a_n = \dfrac{n}{\sqrt{n+1}}$ is not bounded: $a_n = \dfrac{\sqrt{n^2}}{\sqrt{n+1}} = \sqrt{\dfrac{n^2}{n+1}} = \sqrt{\dfrac{n}{1+1/n}} \longrightarrow \infty$,

so it is not bounded. It is monotonic: $f(x) = \dfrac{x}{(x+1)^{1/2}} \Longrightarrow f'(x) = \dfrac{1+(x/2)}{(x+1)^{3/2}} > 0$

so $f(x)$ is increasing, hence $a_n = f(n)$ is monotonic

(b) The function $f(x)$, whose graph is shown, has the Taylor polynomial of degree 2 centered at 0 given by $p_2(x) = a + bx + cx^2$. What can you say about a, b, c? (**Circle the correct answer for each part**) [3]



(i) a is: negative, zero or positive

(ii) b is: negative, zero or positive

(iii) c is: negative, zero or positive

$$f(x) \approx p_2(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2$$

$a = f(0) > 0$ positive; $b = f'(0) > 0$: $f(x)$ increasing at 0; $c = \dfrac{f''(0)}{2} < 0$: concave down

(c) The value of the telescoping series $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)$ is [3]

(i) 0 (ii) 1 (iii) 2 (iv) 1/2 (v) None of the above

Let's first look at one partial sum:

$$s_4 = \sum_{k=1}^{4} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right)$$

Second term in each difference cancels with first term in next difference, so $s_4 = 1 - \frac{1}{5}$.

For general partial sum,

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1}.$$

Thus $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$. Answer is (ii).

3. Consider the following series, all of which converge. For which of these series do you get a conclusive answer when using the **Ratio Test** to check for convergence? Write the letters of all possible answers. If no series satisfies this condition, write"none". You do not need to show your work. [8]

A $\sum_{k=1}^{\infty} \frac{k^6}{k!}$ **B** $\sum_{k=1}^{\infty} \frac{1}{(3k+4)^k}$ **C** $\sum_{k=2}^{\infty} \frac{\ln k}{k^2}$ **D** $\sum_{k=0}^{\infty} (-1)^k \frac{2}{5^k}$

3A $\sum_{k=1}^{\infty} \frac{k^6}{k!}$

$$r = \lim_{k\to\infty} \frac{\frac{(k+1)^6}{(k+1)!}}{\frac{k^6}{k!}} = \lim_{k\to\infty} \frac{(k+1)^6 k!}{k^6 (k+1)! k^6} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^6 \frac{1}{k+1} = 0 < 1:$$ **converges** by Ratio Test.

3B $\sum_{k=1}^{\infty} \frac{1}{(3k+4)^k}$ $r = 0 < 1$

$$r = \lim_{k\to\infty} \frac{1/(3(k+1)+4)^{k+1}}{1/(3k+4)^k} = \lim_{k\to\infty} \frac{(3k+4)^k}{(3k+7)^{k+1}} = \lim_{k\to\infty} \frac{(3k+4)^k}{(3k+7)^k} \frac{1}{3k+7}$$

$$= \lim_{k\to\infty} \left(\frac{3k+4}{3k+7}\right)^k \frac{1}{3k+7} = \lim_{k\to\infty} \left(\frac{1+(4/3)/k}{1+(7/3)/k}\right)^k \frac{1}{3k+7} = \frac{e^{4/3}}{e^{7/3}} \lim_{k\to\infty} \frac{1}{(3k+7)}$$

This has limit 0, which is less than 1, so the series **converges** by the Ratio Test.

3C $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$

$$\lim_{k\to\infty} \frac{\frac{\ln(k+1)}{(k+1)^2}}{\frac{\ln k}{k^2}} = \lim_{k\to\infty} \frac{k^2}{(k+1)^2} \frac{\ln(k+1)}{\ln k} = \lim_{k\to\infty} \frac{k^2}{k^2+2k+1} \frac{\ln(k+1)}{\ln k}$$

$$= \lim_{k\to\infty} \frac{1}{1+2/k+1/k^2}\frac{\ln(k+1)}{\ln k} = \lim_{k\to\infty}\frac{\ln(k+1)}{\ln k}$$

Using L'Hospital's rule, $\displaystyle\lim_{x\to\infty}\frac{\ln(x+1)}{\ln x} = \lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{x}{x+1} = 1$, so

$\displaystyle\lim_{k\to\infty}\frac{\ln(k+1)}{\ln k} = 1$. Thus Ratio Test is **inconclusive**.

3D $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{2}{5^k}$

Here $a_k = (-1)^k\frac{2}{5^k}$, so $\displaystyle\left|\frac{a_{k+1}}{a_k}\right| = \frac{2/5^{k+1}}{2/5^k} = \frac{2\cdot 5^k}{2\cdot 5^{k+1}} = \frac{1}{5}$. Thus $\displaystyle\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{1}{5} = \frac{1}{5} < 1$, so series **converges** by the Ratio Test.

4. Consider the following **sequences** and answer the questions that follow by circling all that apply.

(i) $a_n = \left(\dfrac{2n-1}{n+1}\right)^2$ (ii) $b_n = \dfrac{3^{n+5}}{2^n}$ (iii) $c_n = \sin(n)$ (iv) $S_n = \displaystyle\sum_{k=2}^{n}\frac{k}{k^3-2}$

(a) Which of the above sequences are bounded? (i) a_n (ii) b_n (iii) c_n (iv) S_n [3]

(b) Which of the above sequences are increasing? (i) a_n (ii) b_n (iii) c_n (iv) S_n [3]

(c) Which of the above sequences are convergent? (i) a_n (ii) b_n (iii) c_n (iv) S_n [3]

(i) $a_n = \left(\dfrac{2n-1}{n+1}\right)^2$: we'll show it is bounded, increasing, and convergent.

$$a_n = \frac{4n^2-4n+1}{n^2+2n+1} = \frac{4-4/n+1/n^2}{1+2/n+1/n^2} \longrightarrow 4 \quad \text{as} \quad n\to\infty$$

This sequence is convergent, hence also bounded.

To see if it is increasing, consider $f(x) = \left(\dfrac{2x-1}{x+1}\right)^2$ and look at its derivative for $x \geq 1$.

Using the quotient rule, $f'(x) = 2\left(\dfrac{2x-1}{x+1}\right)\left(\dfrac{2x-1}{x+1}\right)' = 2\left(\dfrac{2x-1}{x+1}\right)\dfrac{3}{(x+1)^2} > 0$ for $x \geq 1$.
$f(x)$ is increasing, so $f(n) = a_n$ is also increasing.

(ii) $b_n = 3^{n+5}/2^n$: we'll show it is not bounded, is increasing, and is not convergent.

$$b_n = \frac{3^{n+5}}{2^n} = 3^5\,\frac{3^n}{2^n} = 3^5\left(\frac{3}{2}\right)^n$$

It is increasing since each term is $\dfrac{3}{2}$ times the previous term. It diverges to ∞,

so it is not bounded.

(iii) $c_n = \sin(n)$: we'll show it is bounded, not increasing, and not convergent.
Since $|c_n| = |\sin(n)| \leq 1$, it is bounded.

The values oscillate often between being ≤ -0.5 and being ≥ 0.5, so it is neither increasing nor convergent.

(iv) $S_n = \displaystyle\sum_{k=2}^{n} \frac{k}{k^3-2}$: we'll show it is bounded, increasing, and convergent.
$S_{n+1} = S_n + \dfrac{n+1}{(n+1)^3-2} > S_n$, so sequence S_n is increasing.

Use Limit Comparison Test on the series $\displaystyle\sum_{k=2}^{\infty} \frac{k}{k^3-2}$. Let's first check $\dfrac{k}{k^3-2} \sim \dfrac{1}{k^2}$:

Both sequences are positive, and $\displaystyle\lim_{k\to\infty} \frac{\frac{1}{k^2}}{\frac{k}{k^3-2}} = \lim_{k\to\infty} \frac{k^3-2}{k^3} = \lim_{k\to\infty} 1 - \frac{2}{k^3} = 1.$

Thus we can compare to the p-series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2}$, which is convergent.

So $\displaystyle\lim_{n\to\infty} S_n = \sum_{k=2}^{\infty} \frac{k}{k^3-2}$ converges by Limit Comp. Test. Thus S_n is convergent and bounded.

5. (a) Find the 3rd-degree Taylor polynomial $T_3(x)$ for $\cos x$ at $a = \dfrac{\pi}{4}$. [4]

$f(x) = \cos x; \quad f'(x) = -\sin x; \quad f''(x) = -\cos x; \quad f'''(x) = \sin x$

$$f(\pi/4) = \frac{1}{\sqrt{2}}; \quad f'(\pi/4) = -\frac{1}{\sqrt{2}}; \quad f''(\pi/4) = -\frac{1}{\sqrt{2}}; \quad f'''(\pi/4) = \frac{1}{\sqrt{2}}$$

$$T_3(x) = f(\pi/4) + f'(\pi/4)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{f'''(\pi/4)}{3!}\left(x - \frac{\pi}{4}\right)^3$$
$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3$$

(b) Use the 3rd-degree Taylor polynomial from part (a) to approximate $\cos\left(\dfrac{3\pi}{16}\right)$. [4]

$$T_3\left(\frac{3\pi}{16}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(-\frac{\pi}{16}\right) - \frac{1}{2\sqrt{2}}\left(-\frac{\pi}{16}\right)^2 + \frac{1}{6\sqrt{2}}\left(-\frac{\pi}{16}\right)^3$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\frac{\pi}{16}\right) - \frac{1}{2\sqrt{2}}\left(\frac{\pi}{16}\right)^2 - \frac{1}{6\sqrt{2}}\left(\frac{\pi}{16}\right)^3 \approx .831424.$$

(c) Estimate the remainder for part b using the Taylor's inequality. [3]

(**Note:** If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ for the Taylor series satisfies the inequality $|R_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$.)

$|\text{error}| \le \frac{M}{4!}\left(x-\frac{\pi}{4}\right)^4$, where $|f^{(4)}(x)| \le M$ for all x such that $\left|x-\frac{\pi}{4}\right| \le \left|\frac{3\pi}{16}-\frac{\pi}{4}\right| = \frac{\pi}{4}$.
From $f'''(x) = \sin x$, get $f^{(4)}(x) = \cos x$, so $|f^{(4)}(x)| \le 1$ for all x. Use $M = 1$: Taylor's inequality says remainder at $x = \frac{3\pi}{16}$ is bounded by $\left|\frac{M}{4!}\left(\frac{\pi}{16}\right)^4\right| \le \frac{\pi^4}{24(16^4)} \approx .0000619$.

6. (a) Determine the Taylor series of $f(x) = e^{-x^2}$ centered at 0. [4]

- First find Taylor Series for e^x:
  $f^{(n)}(x) = e^x \implies f^{(n)}(0) = 1$ for all n, so

$$T(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots + \frac{1}{n!}x^n + \ldots = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

- For e^{-x^2}, replace x by $-x^2$, so Taylor series of e^{-x^2} is

$$1 + (-x^2) + \frac{1}{2!}(-x^2)^2 + \frac{1}{3!}(-x^2)^3 + \frac{1}{4!}(-x^2)^4 + \ldots + \frac{1}{n!}(-x^2)^n + \ldots,$$

which is $1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 - \ldots + (-1)^n\frac{1}{n!}x^{2n} + \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$.

(b) Evaluate $\int e^{-x^2}\,dx$ as an infinite series. (**Remember to include** $+C$) [4]

By (a), $e^{-x^2} = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 - \ldots + (-1)^n\frac{1}{n!}x^{2n} + \ldots$ so integrate termwise:

$$\int e^{-x^2}\,dx = \int 1\,dx - \int x^2\,dx + \int \frac{1}{2!}x^4\,dx - \int \frac{1}{3!}x^6\,dx$$

$$+ \int \frac{1}{4!}x^8\,dx - \ldots + \int (-1)^n\frac{1}{n!}x^{2n}\,dx \ldots + C$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5\cdot 2!}x^5 - \frac{1}{7\cdot 3!}x^7 + \frac{1}{9\cdot 4!}x^9 - \cdots + \frac{(-1)^n}{(2n+1)n!}x^{2n+1} \ldots + C$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)n!}x^{2n+1} + C.$$

(c) Use part (a) to determine the Taylor series of $f(x) = 2xe^{-x^2}$ centered at 0. [4]

We use $e^{-x^2} = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 \ldots + (-1)^n\frac{1}{n!}x^{2n} + \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$

Method 1: Multiply series by $2x$:

$$2xe^{-x^2} = (2x)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n 2}{n!}x^{2n+1}$$

Method 2: Differentiate the series for e^{-x^2} termwise:

$$2xe^{-x^2} = -\left(e^{-x^2}\right)' = -\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}\right)' = -\sum_{n=1}^{\infty} \frac{(-1)^n(2n)}{n!} x^{2n-1}$$

where derivative series starts at $n = 1$ since constant term became 0. Simplifying $\frac{n}{n!}$ to $\frac{1}{(n-1)!}$, $2xe^{-x^2} = -\sum_{n=1}^{\infty} \frac{(-1)^n(2)}{(n-1)!} x^{2n-1} = -\sum_{n=0}^{\infty} \frac{(-1)^{n+1}2}{n!} x^{2(n+1)-1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2}{n!} x^{2n+1}$.

7. Determine whether the following series converge conditionally, converge absolutely or diverge. Show your work in applying any tests used.

(a) $\sum_{k=1}^{\infty} \frac{\sqrt{k^2+1}}{k}$ [4]

For large k, $\frac{\sqrt{k^2+1}}{k} \sim \frac{\sqrt{k^2}}{k} = \frac{k}{k} = 1$, so it looks like the general term tends to 1.

More carefully, $\lim_{k\to\infty} \frac{\sqrt{k^2+1}}{k} = \lim_{k\to\infty} \frac{\sqrt{k^2+1}}{\sqrt{k^2}} = \lim_{k\to\infty} \sqrt{\frac{k^2+1}{k^2}} = \lim_{k\to\infty} \sqrt{\frac{1+1/k^2}{1}} = 1.$

This series **diverges** by the Divergence Test.

(b) $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ [4]

$f(x) = \frac{1}{x \ln x}$ is positive and continuous for $x \geq 2$, and $f'(x) = \frac{-(\ln x + 1)}{(x \ln x)^2} < 0$, so $f(x)$ is decreasing. We use Integral Test: $\int \frac{1}{x \ln x}\, dx \overset{u=\ln x}{=} \int \frac{1}{u}\, du = \ln|u| + C = \ln|\ln x| + C$,

so $\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b\to\infty} \int_2^b \frac{1}{x \ln x}\, dx = \lim_{b\to\infty} (\ln|\ln b| - \ln|\ln 2|) = \infty.$

Therefore the series **diverges** by the Integral Test.

(c) $\sum_{k=0}^{\infty} \frac{4+3^k}{4^k}$ [4]

Break up the kth term into $\frac{4}{4^k} + \frac{3^k}{4^k} = 4\left(\frac{1}{4}\right)^k + \left(\frac{3}{4}\right)^k$. Using geometric series, both $\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k$ and $\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k$ converge, so $\sum_{k=0}^{\infty} \frac{4+3^k}{4^k} = 4\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k + \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k$ converges using geometric series. Terms are positive, so we have **absolute convergence**. (This can also be settled using the Ratio Test, with $r = 3/4 < 1$.)

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{k^4}{e^{3k}}$ [4]

We use the Ratio Test: $\displaystyle\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\frac{(k+1)^4}{e^{3(k+1)}}}{\frac{k^4}{e^{3k}}} = \lim_{k\to\infty}\frac{(k+1)^4}{k^4}\frac{e^{3k}}{e^{3k+3}}$

$\displaystyle = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^4\frac{1}{e^3} = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^4\frac{1}{e^3} = \frac{1}{e^3} < 1$, so $\displaystyle\sum_{k=1}^{\infty}\frac{k^4}{e^{3k}}$ **converges absolutely** using the Ratio Test.

(e) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k^2-1}$ [4]

Check absolute convergence with $\displaystyle\sum_{k=2}^{\infty}\left|\frac{(-1)^k}{k^2-1}\right| = \sum_{k=2}^{\infty}\frac{1}{k^2-1}$. Since $\displaystyle\frac{1}{k^2-1} \sim \frac{1}{k^2}$, we want to compare to $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2}$, which we can do since terms are positive and

$\displaystyle\lim_{k\to\infty}\frac{1/k^2}{1/(k^2-1)} = \lim_{k\to\infty}\frac{k^2-1}{k^2} = \lim_{k\to\infty} 1 - \frac{1}{k^2} = 1$. Since $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2}$ converges (p-series with $p = 2 > 1$),

$\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^k}{k^2-1}$ **converges absolutely** by the Limit Comparison Test.

(f) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k^2+1}}$ [4]

Check absolute convergence with $\displaystyle\sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}}{\sqrt{k^2+1}}\right| = \sum_{k=1}^{\infty}\frac{1}{\sqrt{k^2+1}}$. Since $\displaystyle\frac{1}{\sqrt{k^2+1}} \sim \frac{1}{\sqrt{k^2}} = \frac{1}{k}$ we want to compare to $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$, which we can do since

$$\lim_{k\to\infty}\frac{1/k}{1/\sqrt{k^2+1}} = \lim_{k\to\infty}\frac{\sqrt{k^2+1}}{k} = \lim_{k\to\infty}\frac{\sqrt{k^2+1}}{\sqrt{k^2}} = \lim_{k\to\infty}\sqrt{\frac{k^2+1}{k^2}} = \lim_{k\to\infty}\sqrt{1+\frac{1}{k^2}} = 1.$$

Thus divergence of $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ (harmonic series) implies $\displaystyle\sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}}{\sqrt{k^2+1}}\right|$ diverges by Limit Comparison Test, so no absolute convergence of $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\sqrt{k^2+1}}$.

To decide if series converges conditionally, use Alternating Series Test. The terms $\displaystyle\frac{1}{\sqrt{k^2+1}}$ are positive and decreasing (since reciprocal $\sqrt{k^2+1}$ is clearly increasing – could show

derivative of $1/\sqrt{x^2+1}$ is negative instead) with limit 0 as $k \to \infty$.
Therefore the series **converges conditionally**.

8. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n n}(x-2)^n$. [6]

Step 1. Use Ratio Test on $\displaystyle\sum_{n=1}^{\infty} \left|\frac{1}{2^n n}(x-2)^n\right| = \sum_{n=1}^{\infty} \frac{1}{2^n n}|x-2|^n$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{2^{n+1}(n+1)}|x-2|^{n+1}}{\frac{1}{2^n n}|x-2|^n} = \lim_{n\to\infty}\frac{2^n n}{2^{n+1}(n+1)}|x-2| = \lim_{n\to\infty}\frac{n}{2(n+1)}|x-2|,$$

which is $\frac{1}{2}|x-2|$. The radius of convergence is determined by $\frac{1}{2}|x-2| < 1$, namely $|x-2| < 2$.

Since $|x-2| < 2 \iff -2 < x-2 < 2 \iff 0 < x < 4$, the interval of convergence is $(0,4), [0,4), (0,4]$, or $[0,4]$. We have to check the endpoints.

Step 2. Test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n n}(x-2)^n$ for convergence at $x=0$ and 4.

Take $x=0$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n n}(-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges since it's alternating harmonic series.

Take $x=4$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n n}2^n = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges (harmonic series).

The **interval of convergence** is $[0,4)$.

9. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n}x^n$.

(a) Find the radius of convergence for this series.

Ratio Test on $\displaystyle\sum_{n=1}^{\infty} \left|\frac{3^n}{n}x^n\right| = \sum_{n=1}^{\infty} \frac{3^n}{n}|x|^n$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{3^{n+1}}{n+1}|x|^{n+1}}{\frac{3^n}{n}|x|^n} = \lim_{n\to\infty}\frac{3^{n+1}}{3^n}\frac{n}{n+1}|x| = \lim_{n\to\infty} 3\frac{1}{1+1/n}3|x| = 3|x|.$$

Radius of convergence determined by $3|x| < 1$, so $|x| < \frac{1}{3}$: **radius of convergence** is $\frac{1}{3}$.

(b) Find the interval of convergence.

By (a), interval is $(-1/3, 1/3), [-1/3, 1/3), (-1/3, 1/3]$, or $[-1/3, 1/3]$. Check endpoints $x = \pm\frac{1}{3}$.

Take $x = \frac{1}{3}$: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n}\left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}$. This diverges since it's the harmonic series.

Take $x = -\frac{1}{3}$: $\sum_{n=1}^{\infty} \frac{3^n}{n}\left(-\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges since it's alternating harmonic series. The **interval of convergence** is $[-1/3, 1/3)$.

(c) For which x does this series converge absolutely?

Every power series converges absolutely on the interior of its interval of convergence (that is, away from the endpoints). At the endpoints $\pm 1/3$ the sum of absolute values of terms is the harmonic series both places, so there is not absolute convergence there. The series converges absolutely for x in $(-1/3, 1/3)$.

10. How many terms of the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(n+5)^2}$ do we need to add in order to find the sum of the series to within an accuracy of 0.001 (that is, $|\text{error}| < 0.001$)? [4]

The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(n+5)^2}$ is alternating (and nth term in absolute value is decreasing since the reciprocal $(n+5)^2$ is increasing), so the nth partial sum estimates the full series with an error that is at most the absolute value of the first omitted term, and that is $\frac{1}{((n+1)+5)^2}$.

Therefore to find an n making the nth partial sum within .001 of the full series, we seek an n that makes $\frac{1}{((n+1)+5)^2} < .001 = \frac{1}{10^3}$:

$$\frac{1}{(n+6)^2} < \frac{1}{10^3} \iff (n+6)^2 > 10^3 \iff n+6 > \sqrt{10^3} \iff n > \sqrt{10^3} - 6 \iff n > 25.6$$

Use $n = 26$ (or higher) for the error estimate to guarantee the nth partial sum is within .001 of the series.