Practice Exam 1 Solutions

1

a) T

As $n \to \infty$, also $2n+1 \to \infty$, so $a_{2n+1} \to L$.

b) F Work to stretch spring a length L from natural (rest) length is

$$W = \int_0^L kx\,dx = \frac{kL^2}{2}.$$

Doubling L will multiply W by factor of 4, not 2.

c) F x^3 is concave up, so trapezoids are above graph.

d) F Ex: $\int_0^1 \frac{dx}{\sqrt{x}}$ improper, $\int_1^2 \frac{dx}{\sqrt{x}}$ is not improper

e) T This is the harmonic series

$$1 + \underbrace{\tfrac{1}{2}}_{=\frac{1}{2}} + \underbrace{\tfrac{1}{3} + \tfrac{1}{4}}_{>\frac{2}{4}=\frac{1}{2}} + \underbrace{\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}}_{>\frac{4}{8}=\frac{1}{2}} + \underbrace{\tfrac{1}{9} + \cdots + \tfrac{1}{16}}_{>\frac{8}{16}=\frac{1}{2}} + \cdots$$

Q2 A)

$$\int_1^\infty \frac{dx}{\sqrt{x}} \quad \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}}dx = \lim_{t\to\infty} \int_1^t x^{-1/2}dx$$

$$= \lim_{t\to\infty} \left.\frac{x^{1/2}}{1/2}\right|_1^t$$

$$= \lim_{t\to\infty} 2\sqrt{x}\Big|_1^t = \lim_{t\to\infty} 2\sqrt{t}-2$$

$$\to \infty.$$

$$\therefore \int_1^\infty \frac{dx}{\sqrt{x}} \text{ diverges}$$

Q2 B) $\displaystyle\int_0^\infty \frac{dx}{(2x+1)(x+4)}$ Use partial fractions

$$\frac{1}{(2(x)+1)(x+4)} = \frac{A}{(2x+1)} + \frac{B}{(x+4)} \qquad A(x+4)+B(2x+1) = 1$$

To find A & B

set $x=-4$

$A(0)+B(-7)=1$

$B = -1/7$

set $x=-1/2$

$A\cdot\frac{7}{2}=1 \quad A=2/7$

$$\int_0^\infty \frac{dx}{(2x+1)(x+4)} = \int_0^\infty \left[\frac{2/7}{(2x+1)} - \frac{1/7}{(x+4)}\right] dx$$

$$\lim_{t\to\infty}\left[\int_0^t \left(\frac{2/7}{(2x+1)} - \frac{1/7}{(x+4)}\right) dx\right]$$

$$= \lim_{t\to\infty}\left[\frac{2/7\cdot \ln|2x+1|}{2} - 1/7\ln|x+4|\right]_0^t$$

$$= \frac{1}{7}\left[\lim_{t\to\infty}\Big[\ln|2x+1| - \ln|x+4|\Big]_0^t\right]$$

$$= \frac{1}{7}\left[\lim_{t\to\infty} \ln\left|\frac{2x+1}{x+4}\right|\right]_0^t$$

$$= \frac{1}{7}\lim_{t\to\infty}\left[\ln\left|\frac{2t+1}{t+4}\right| - \ln\left|\frac{1}{4}\right|\right]$$

$$= \frac{1}{7}\ln|2| - \ln\left|\frac{1}{4}\right| = \frac{1}{7}\ln\left|\frac{2}{1/4}\right| = \frac{1}{7}\ln|8| = \frac{3}{7}\ln|2|$$

$\therefore \int_0^\infty \frac{dx}{(2x+1)(x+4)} \to$ converges.

Q2c] $\int_1^\infty \frac{\ln x}{x}dx$ $\quad \int \frac{\ln x}{x}dx \quad u = \ln x \Rightarrow du = \frac{dx}{x}$

$$\int u\,du = \frac{u^2}{2} + c = \frac{(\ln x)^2}{2} + c$$

$$\lim_{t\to\infty}\int_1^t \frac{\ln x}{x}dx = \lim_{t\to\infty} \left.\frac{(\ln x)^2}{2}\right|_1^t$$

$$= \lim_{t\to\infty}\left[\frac{\ln(t)^2}{2} - \cancel{\frac{\ln(1)^2}{2}}^{\,0}\right]$$

$\Rightarrow \infty$ as $t\to\infty$

$\therefore \int_1^\infty \frac{\ln x}{x}dx \to$ divergent

Q3 A) $\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64} \cdots \right\}$

$$a_n = \frac{1}{2^n}$$

Q3 B) $\left\{ \frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, \frac{-16}{5}, \frac{25}{6}, -\frac{36}{7}, \ldots \right\}$

The numerator of nth term up to sign is n^2, donominator is $(n+1)$. It is alternating in sign:

$$a_n = (-1)^{n+1} \frac{n^2}{n+1} = (-1)^{n-1} \frac{n^2}{n+1}$$

Q4A) $a_n = \frac{n^4}{n^3 - 2n}$ To test if the sequence converges or diverges we need to check the end behavior

$\Rightarrow \lim\limits_{n \to \infty} \frac{n^4}{n^3 - 2n}$ is what interests us.

Divide the num & deno by highest power of 'n' in deno.

$$\lim_{n \to \infty} \frac{n^4/n^3}{\frac{n^3}{n^3} - \frac{2n}{n^3}} = \lim_{n \to \infty} \frac{n}{1 - \cancelto{0}{2/n^2}} \to \infty \text{ as } n \to \infty$$

$\therefore$ $a_n = \frac{n^4}{n^3 - 2n}$ diverges.

Q4B] $a_n = \frac{\cos^2 n}{2^n}$

since $0 \le \cos^2 n \le 1$

$$0 \le \frac{\cos^2 n}{2^n} \le \frac{1}{2^n}$$

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{\cos^2 n}{2^n} \le \lim_{n\to\infty} \cancelto{0}{\frac{1}{2^n}}$$

$\therefore$ By squeeze theorem $\lim_{n\to\infty} \frac{\cos^2 n}{2^n} = 0$

$\therefore$ $a_n = \frac{\cos^2 n}{2^n}$ converges to 0.

Q5 A] $\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$

Using test of divergence: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2+n}{1-2n}$

$$= -\frac{1}{2} \neq 0$$

$\therefore$ By test of divergence $\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$ diverges.

Q5b. $\sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n} = \sum_{n=1}^{\infty}\left(\left(\frac{2}{e}\right)^n+\left(\frac{4}{e}\right)^n\right)$

$\lim_{n\to\infty} \left(\frac{2}{e}\right)^n+\left(\frac{4}{e}\right)^n = ?$

$\left(\frac{2}{e}\right)^n \to 0$ as $n\to\infty$, $\left(\frac{4}{e}\right)^n \to \infty$ as $n \to \infty$.

$\therefore \lim_{n\to\infty}\left(\frac{2^n+4^n}{e^n}\right) \to \infty$

$\therefore \sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n} \to$ diverges.

$\boxed{\text{Q5 c} \rightarrow \text{Not on the exam}}$

Q5 d $\sum_{n=2}^{\infty} \frac{1}{n^2-n} = \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ use partial fractions decomposition

$\frac{1}{n(n-1)} = \frac{A}{n} + \frac{B}{(n-1)}$ $\quad A(n-1)+B(n) = 1$

To find A & B
$n=1 \quad B=1$
$n=0 \quad A=-1$

$\sum_{n=2}^{\infty}\left(\frac{-1}{n}+\frac{1}{n-1}\right) = \sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right)$

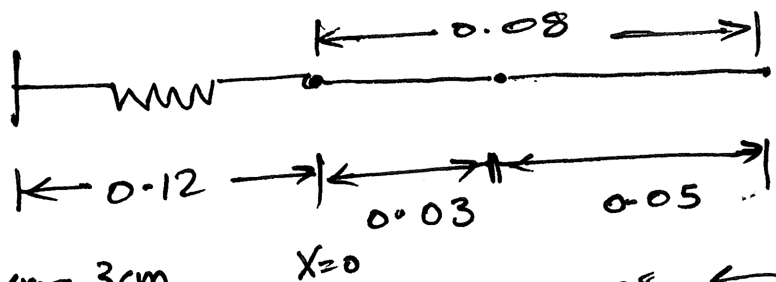
$\Rightarrow S_n = \left(1-\cancel{\tfrac{1}{2}}\right)+\left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right)+\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{4}}\right)+\cdots+\left(\cancel{\tfrac{1}{n-2}}-\cancel{\tfrac{1}{n-1}}\right)+\left(\cancel{\tfrac{1}{n-1}}-\tfrac{1}{n}\right)$

$S_n = 1-\frac{1}{n} \Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1-\cancel{\frac{1}{n}}^{\,0} = 1$

$\sum_{n=2}^{\infty} \frac{1}{n^2-n}$ converges & the sum is 1

(Example of a telescoping series)

Q6A)



$15cm - 12cm = 3cm$
$= .03m$

$20cm - 12cm = 8cm$
$= .08m$

$F = KX$

$\Rightarrow 30 = K.(0.03)$

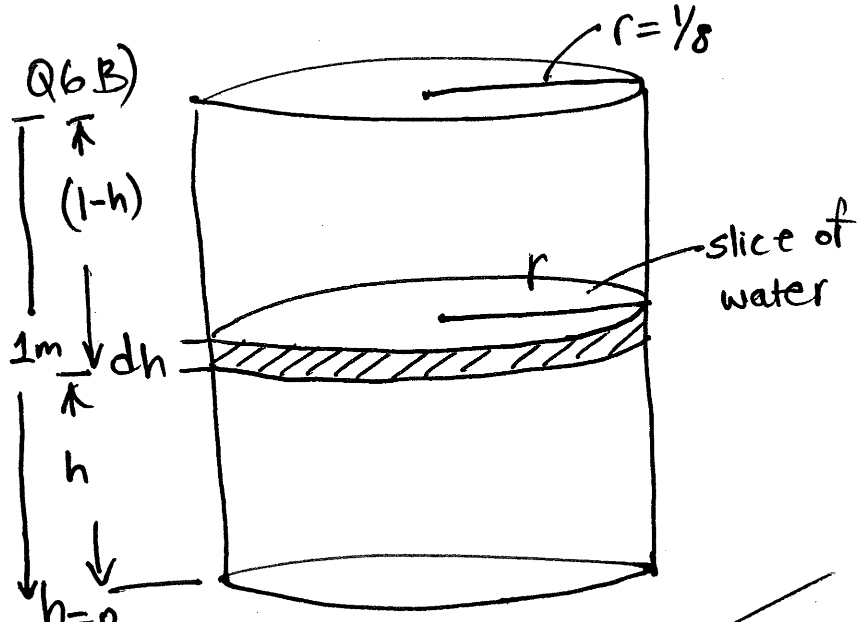
$\Rightarrow K = 1000$

$$W = \int_0^{0.08} K.x.dx$$

$$= \frac{1}{2}Kx^2 \Big]_0^{0.08}$$

$$= \frac{1}{2}1000(0.08)^2 = 3.2 \text{ Joules}$$

Q6B)



$A_s = \pi r^2 = \pi.\left(\frac{1}{8}\right)^2 = \frac{\pi}{64}$

$V_s = \frac{\pi}{64}.dh$

$m = V_s.\text{density}$

$= \frac{\pi}{64}dh.2000 kg/m^3$

$F = m.\text{gravity} = m.g$

$\because (g = 9.81)$

$F_s = \frac{\pi}{64}.2000(9.81).dh$

W_{slice} = Work done in moving the slice of beer out of the top of the keg

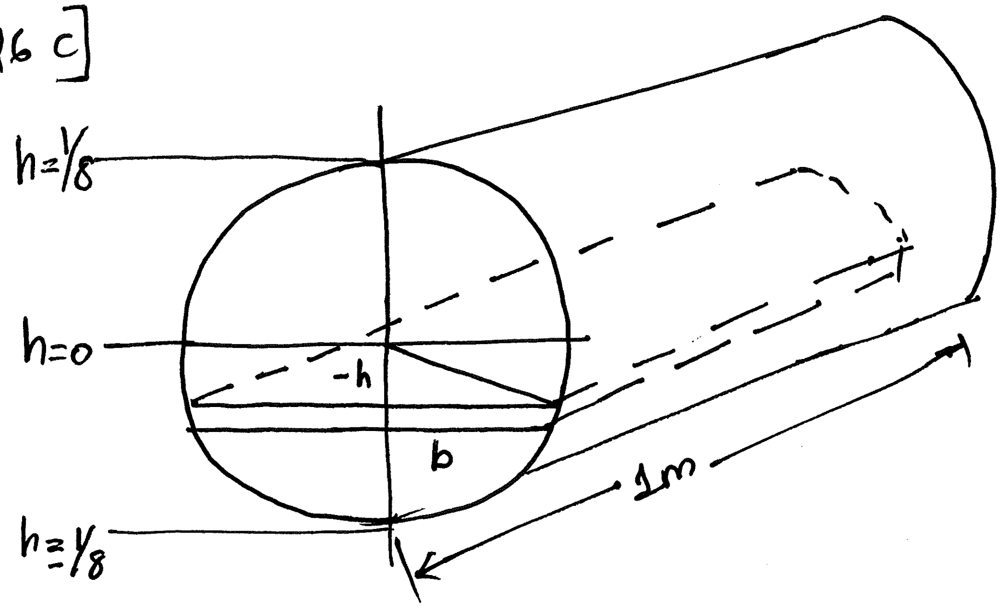
$W_s = F_s.\text{dist}$

$\text{dist} = (1-h)$

$\therefore W_s = \frac{\pi}{64}2000(9.81).(1-h).dh$

$$W_{Total} = \int_0^{1/2} \frac{\pi}{64}2000(9.81)(1-h)dh$$

Q6 c]



$$(-h)^2 + b^2 = \left(\frac{1}{8}\right)^2$$

$$h^2 + b^2 = \frac{1}{64}$$

$$b = \frac{1}{8}\sqrt{1-64h^2}$$

$$\text{Width} = 2b$$

$$= 2 \cdot \frac{1}{8}\sqrt{1-64h^2}$$

$$= (1/4)\sqrt{1-64h^2}$$

$$A_{slice} = \text{width} \cdot 1$$

$$= \frac{1}{4}\sqrt{1-64h^2}$$

$$V_s = \frac{1}{4}\sqrt{1-64h^2}\,.dh$$

$$m = \frac{1}{\cancel{4}}\sqrt{1-64h^2}\,.\,\overset{500}{\cancel{2000}}\,.dh$$

$$m = 500\sqrt{1-64h^2}\,.dh$$

$$F = 500\sqrt{1-64h^2}\,.dh.(9.81)$$

$$W_s = 500\sqrt{1-64h^2}\,dh.(9.81).(1/8 - h)$$

$$W_{Total} = \int_{-1/8}^{0} 500(9.81)\sqrt{1-64h^2}\,(1/8-h)\,dh$$

Q7 $\displaystyle\int_0^1 \frac{1}{3^x}\,dx$ Goal is to find 'n' such that the trapezoid rule approximates the integral within .001 error.

First.

$$\left| \int_a^b f(x)\,dx - T_n \right| \le \frac{k(b-a)^3}{12n^2}$$

K is the upper bound on $|f''(x)|$ over $[a,b]$

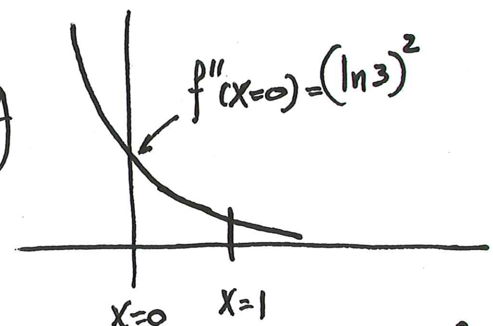
$\therefore\ |f''(x)| \le K$ for $a \le x \le b$

start with finding $f''(x)$

$$f(x) = \frac{1}{3^x}$$

$$f(x) = 3^{-x} \quad f'(x) = -3^{-x}\ln(3) \quad f''(x) = 3^{-x}(\ln 3)^2 = \frac{(\ln(3))^2}{3^x}$$

$f''(x)$ is decreasing



So the maximum value of $K = (\ln 3)^2$

$$\text{Error} \le \frac{K(b-a)^3}{12n^2} = \frac{(\ln 3)^2(1-0)^2}{12n^2} \le 0.001$$

$$\Leftrightarrow \frac{(\ln(3))^2}{12n^2} \le .001 \Leftrightarrow n^2 \geq \frac{[\ln(3)]^2}{12\cdot(.001)}$$

$$\Leftrightarrow n^2 \geq 100.57 \Leftrightarrow n \geq 10.02$$

So $\boxed{n \geq 11}$ Assures us that error will be less than .001 using T_n.

8 a) $[a,b] = [1,3]$. $\boxed{f(x) = \frac{1}{e^x}}$

For T_4, $\Delta x = \frac{b-a}{4} = \frac{2}{4} = \frac{1}{2}$

1	1.5	2	2.5	3
x_0	x_1	x_2	x_3	x_4

$$T_4 = \left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right)\frac{\Delta x}{2}$$

$$= \left(f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)\right)\frac{1}{4}$$

$$= .32469$$

For M_8, $\Delta x = \frac{b-a}{8} = \frac{2}{8} = \frac{1}{4}$

$\bar{x}_2 = \frac{1.25+1.5}{2} = 1.375$

1.25, 1.75, 2.25, 2.75

$x_0 = 1$, 1.5, 2, 2.5, $3 = x_8$

$\bar{x}_1 = \frac{1+1.25}{2} = 1.125$

$$M_8 = \left(f(\bar{x}_1) + \ldots + f(\bar{x}_8)\right)\Delta x$$
$$= \left(f(\bar{x}_1) + \ldots + f(\bar{x}_8)\right)\frac{1}{4}$$

n	1	2	3	4	5	6	7	8
$\bar{x}_n$	1.125	1.375	1.625	1.875	2.125	2.375	2.625	2.875

$\bar{x}_n = \frac{x_{n-1}+x_n}{2}$

$$= .31726$$

b) $f(x) = \frac{1}{e^x} = e^{-x} \Rightarrow f'(x) = -e^{-x} \Rightarrow f''(x) = -(-e^{-x}) = e^{-x}$

For $1 \le x \le 3$, $|f''(x)| = \frac{1}{e^x} \le \frac{1}{e^1} = \frac{1}{e} \Rightarrow$ use $K = \frac{1}{e}$.

Error for M_4 is <u>at most</u> $\frac{K(b-a)^3}{24n^2} = \frac{\frac{1}{e}\cdot 2^3}{24\cdot 4^2} = .0076\ldots$

Error for T_8 is <u>at most</u> $\frac{K(b-a)^3}{12n^2} = \frac{\frac{1}{e}\cdot 2^3}{12\cdot 8^2} = .0038\ldots$

<u>Remark</u> Actual error for M_4 is .0065
Actual error for T_8 is .00082

Not based on approx. integration, but <u>exact</u> calculation $\int_1^3 \frac{1}{e^x}dx = \frac{1}{e} - \frac{1}{e^3}$

c) To get $\underline{T_n\text{-error}}$ $\leq .0001 = \frac{1}{10^4}$, want

$$\frac{K(b-a)^3}{12n^2} \leq \frac{1}{10^4}$$

$$\frac{\frac{1}{e}\cdot 2^3}{12n^2} \leq \frac{1}{10^4}$$

$$n^2 \geq \frac{10^4\cdot\frac{8}{e}}{12} \approx 2452.5$$

$$n \geq 49.5 \longrightarrow \underline{n \geq 50}$$

To $\underline{\text{get } M_n\text{-error}}$ $\leq .0001 = \frac{1}{10^4}$, want

$$\frac{K(b-a)^3}{24n^2} \leq \frac{1}{10^4}$$

$$\frac{\frac{1}{e}\cdot 2^3}{24n^2} \leq \frac{1}{10^4}$$

$$n^2 \geq \frac{10^4\cdot\frac{8}{e}}{24} \approx 1226.2$$

$$n \geq 35.01 \longrightarrow \underline{n \geq 35}$$