Important Notice: To prepare for the final exam, study past exams and practice exams, and homeworks, quizzes, and worksheets, not just this practice final. A topic not being on the practice final does not mean it won't appear on the final.

1. If the statement is always true, circle the printed capital T. If the statement is sometimes false, circle the printed capital F. In each case, write a careful and clear justification or a counterexample.

   (a) If a force of $F(x) = 6x$ pounds is required to stretch a spring x feet beyond its rest length, then 36 ft-lbs of work is done in stretching the spring from its natural length to 6 feet beyond its rest length. F

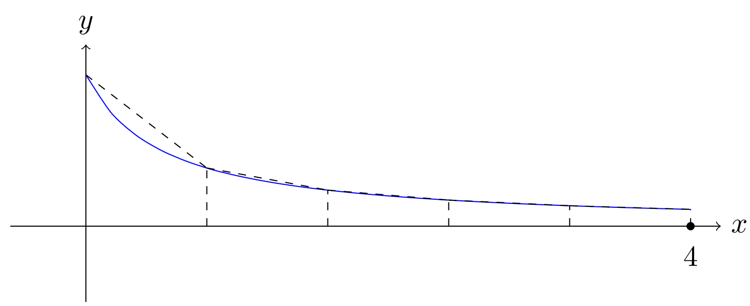
   Work: $W = \int_a^b F(x)\,dx$

   Work done stretching the spring from $x = 0$ to $x = 6$:

   $$W = \int_0^6 6x\,dx = \left. 3x^2 \right|_0^6 = 108 \text{ ft-lbs}$$

   (b) The trapezoid rule with $n = 5$ for $\int_0^4 \frac{dx}{2x+1}$ will be an overestimate. T

   Justification: The curve $y = 1/(2x+1)$ is decreasing and concave up, so the trapezoid tops lie above the graph throughout. The estimate is larger than the integral.



   (c) $\ln(2.5) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1.5)^n}{n}$. F

   We have

   $$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

   with radius of convergence 1; the series on the right diverges for $|x| > 1$. Therefore when we set $x = 1.5$, the left side is $\ln(2.5)$ but the right side is divergent.

(d) The improper integral $\displaystyle\int_1^\infty \frac{x^2}{(x^3+7)^{1/3}}\,dx$ converges. F

$$\int_1^\infty \frac{x^2}{(x^3+7)^{1/3}}\,dx = \lim_{b\to\infty}\int_1^b \frac{x^2}{(x^3+7)^{1/3}}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{2}(x^3+7)^{2/3}\Big|_1^b$$

$$= \lim_{b\to\infty}\left(\frac{1}{2}(b^3+7)^{2/3} - \frac{1}{2}(8)^{2/3}\right) = \infty$$

This limit does not exist so $\displaystyle\int_1^\infty \frac{x^2}{(x^3+7)^{1/3}}\,dx$ does not converge.

(e) The tangent line to the parametric curve $(x,y) = (t - 1/t, 4 + t^2)$ at the point where $t = 1$ has equation $y = x + 5$. T

Slope $\displaystyle\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1 + 1/t^2}$

$\displaystyle\frac{dy}{dx} = \frac{2}{2} = 1$ when $t = 1$

The tangent line is $y = x + b$ and passes through $(x,y) = (0,5)$

The line is $y = x + 5$

2. For each multiple choice question, circle the correct answer. There is only one correct choice for each answer.

(a) A cylindrical tank with a radius of 1 meter and a height of 8 meters is half full. Letting $y = 0$ correspond to the top of the tank, the density of water be 1000 kg/m^3, and g be the acceleration due to gravity in m/sec^2, the work required to pump the water out of the tank is

(i) $1000\pi g\displaystyle\int_4^8 y\,dy$ (ii) $1000\pi g\displaystyle\int_0^8 y\,dy$ (iii) $1000\pi g\displaystyle\int_0^4 y\,dy$ (iv) $16000\pi g\displaystyle\int_4^8 y\,dy$

The y-values for the water in the tank are $4 \le y \le 8$. Partition $[4,8]$ into n parts. For $k = 1, \ldots, n$,

$V_k = \pi r^2 \Delta y = \pi \Delta y$ weight $= \pi\Delta y \cdot 1000g$

k-th slice is lifted y_k meters (since $y = 0$ is the top of the tank).

$W_k \approx 1000g\pi\, y_k \Delta y$ $\displaystyle\sum_{k=1}^n W_k \approx \sum_{k=1}^n 1000g\pi\, y_k\,\Delta y$

$W = 1000\pi g\displaystyle\int_4^8 y\,dy$, which is (i). It is also $1000\pi g\displaystyle\int_0^4 (8-y)\,dy$.

(b) The Taylor series at $x = 0$ for $\sin x$ is

(i) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{n!}$ (ii) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ (iii) $\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-1}}{(2n-1)!}$ (iv) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$

The Taylor series at $x = a$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

Let $f(x) = \sin x$ and $a = 0$. Then

- $f(0) = f^{(4)}(0) = \sin 0 = 0$
- $f'(0) = f^{(5)}(0) = \cos 0 = 1$
- $f''(0) = f^{(6)}(0) = -\sin 0 = 0$
- $f^{(3)}(0) = f^{(7)}(0) = -\cos 0 = -1$

The Taylor series for $\sin x$ at 0 is $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$, which is (iv). (It is not (iii), which is off by a minus sign.)

(c) A parametric curve tracing out the circle once clockwise for $0 \leq t \leq \pi$ starting at $(1, 0)$ is

(i) $(\cos t, \sin t)$ traces out the top half of the circle counter-clockwise starting at $(1, 0)$.

(ii) $(\cos t, -\sin t)$ traces out the bottom half of the circle clockwise starting at $(1, 0)$.

(iii) $(\cos(2t), \sin(2t))$ traces out the complete circle counter-clockwise starting at $(1, 0)$.

(iv) $(\cos(2t), -\sin(2t))$ traces out the complete circle clockwise starting at $(1, 0)$.

The answer is (iv).

3. This question is about the curve $y = \tan x$.

(a) Write a definite integral that gives the arc length of curve $y = \tan x$ from $x = 0$ to $x = \pi/4$.

Since $dy/dx = \sec^2 x$, the arc length is $\int_0^{\pi/4} \sqrt{1 + (dy/dx)^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \sec^4 x}\, dx$.

(b) Write a definite integral that gives the area of the surface formed by revolving the curve $y = \tan x$ from $x = 0$ to $x = \pi/4$ around the x-axis.

The surface area is $\int_0^{\pi/4} 2\pi y\sqrt{1 + (dy/dx)^2}\, dx = \int_0^{\pi/4} 2\pi \tan x\sqrt{1 + \sec^4 x}\, dx$

4. Use the error bound formulas on the last page to determine an n such that the Trapezoid rule with n subintervals approximates $\int_0^1 \frac{1}{e^x}\, dx$ to within .001.

By the last page, $|\text{error}| \leq \dfrac{K(b-a)^3}{12n^2}$, where $|f''(x)| \leq K$ for all x in $[a,b]$ with $a=0$, $b=1$, $f(x)=e^{-x}$.

$f(x)=e^{-x} \Longrightarrow f'(x)=-e^{-x} \Longrightarrow f''(x)=e^{-x}$

Since $f''(x)$ is decreasing, on $[0,1]$ we have $|f''(x)|=e^{-x} \leq f''(0)=1$, so we can use $K=1$. Thus

$|\text{error}| \leq \dfrac{K(b-a)^3}{12n^2} = \dfrac{1(1^3)}{12n^2} \overset{?}{\leq} 0.001 = \dfrac{1}{1000}$

Solve for n: $n^2 \geq \dfrac{1000}{12} \approx 83.333$. The least n we can choose by this method is $n=10$.

5. (a) Obtain the Taylor series for $\dfrac{1}{1+x}$ at $x=0$ from the geometric series for $\dfrac{1}{1-x}$, writing your final answer in summation notation.

- $\dfrac{1}{1-x} = 1+x+x^2+x^3+\cdots = \displaystyle\sum_{n=0}^{\infty} x^n$ for $|x|<1$.
- Replacing x with $-x$, $\dfrac{1}{1+x} = 1-x+x^2-x^3+\cdots = \displaystyle\sum_{n=0}^{\infty}(-1)^n x^n$ for $|x|<1$.

(b) Use your result from part (a) and integration to write down the Taylor series at $x=0$ for $\ln(1+x)$ and then find the radius of convergence of that series.

Integrating termwise, $\displaystyle\int \frac{1}{1+x}\,dx = \int 1\,dx - \int x\,dx + \int x^2\,dx - \int x^3\,dx + \ldots$, so

$$\ln(1+x) + C = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}.$$

To find C, let $x=0$: $\ln(1)+C=0 \Longrightarrow C=0$. Thus

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}.$$

The radius of convergence comes from the ratio test: when $a_n = (-1)^{n-1}x^n/n$, $|a_{n+1}/a_n| = (n/(n+1))|x| \to |x|$ as $n\to\infty$, so the radius of convergence is 1.

6. (a) Find the 3rd-order Taylor polynomial centered at 4 for $\dfrac{1}{\sqrt{x}}$.

$$T_3(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3$$

- $f(x) = \dfrac{1}{\sqrt{x}} = x^{-1/2}$
- $f'(x) = -\dfrac{1}{2}x^{-3/2} = -\dfrac{1}{2x^{3/2}}$
- $f''(x) = \dfrac{3}{4}x^{-5/2} = \dfrac{3}{4x^{5/2}}$
- $f'''(x) = -\dfrac{15}{8}x^{-7/2} = -\dfrac{15}{8x^{7/2}}$
- $f(4) = \dfrac{1}{2}$
- $f'(4) = -\dfrac{1}{2(2^3)} = -\dfrac{1}{16}$
- $f''(4) = \dfrac{3}{4(2^5)} = \dfrac{3}{128} \Rightarrow \dfrac{f''(4)}{2} = \dfrac{3}{256}$
- $f'''(4) = -\dfrac{15}{8(2^7)} \Rightarrow \dfrac{f'''(4)}{6} = -\dfrac{5}{2048}$

Thus
$$T_3(x) = \frac{1}{2} - \frac{1}{16}(x-4) + \frac{3}{256}(x-4)^2 - \frac{5}{2048}(x-4)^3.$$

(b) Use Taylor's inequality (see the last page) to give an upper bound for the error in approximating $\dfrac{1}{\sqrt{3.99}}$ by the polynomial in part (a) at $x = 3.99$.

We need an M for which $|f^{(4)}(x)| \le M$ when $|x-4| \le |3.99-4| = .01$. Then $|\text{error}| \le \dfrac{M}{4!}|3.99-4|^4$

- $f^{(4)}(x) = \dfrac{105}{16}x^{-9/2} = \dfrac{105}{16x^{9/2}}$, which is a **decreasing function** for $x > 0$
- For $|x-4| \le |3.99-4|$, $|f^{(4)}(x)| \le |f^{(4)}(3.99)| = \dfrac{105}{16(3.99)^{9/2}} \approx .013$. We can use $M = .013$.
- $|\text{error}| \le \dfrac{.013}{4!}|3.99-4|^4 = 5.41 \times 10^{-12}$. (The exact error is $|T_3(3.99) - 1/\sqrt{3.99}| \approx 5.35 \times 10^{-12}$, so the upper bound on the error is quite sharp.)

7. Use the error bound for alternating series to determine how many terms of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ need to be added to estimate the full series with $|\text{error}| < 0.001$.

The series is alternating (signs are alternating) and the magnitude of the terms $1/(n^2+1)$ is decreasing to 0 since n^2+1 is increasing to ∞. Therefore the error in approximating the series using the nth partial sum is at most the magnitude of the first omitted term. So if we use the nth partial sum then the error is $< \dfrac{1}{(n+1)^2+1}$. We seek n such that $\dfrac{1}{(n+1)^2+1} < 0.001 = \dfrac{1}{1000}$.

Since $(n+1)^2 + 1 > 1000 \iff (n+1)^2 > 999 \iff n+1 > 31.6 \iff n > 30.6$. We can use $n \ge 31$. Thus the sum of the first 31 terms approximates the full series with the desired error.

8. Use the Integral Test to show $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p < 1$.

Let $f(x) = \dfrac{1}{x^p} = x^{-p}$. This is continuous and positive. Also

- $f'(x) = -px^{-p-1} = -\dfrac{p}{x^{p+1}} < 0$ so $f(x)$ decreases and
- $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{1}{x^p} = 0$.

Suppose $p > 1$.

- $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty} \frac{x^{1-p}}{1-p}\bigg|_0^b = \lim_{b\to\infty} \frac{b^{1-p}}{1-p} - \frac{1}{1-p} = \lim_{b\to\infty} \frac{1}{(1-p)b^{p-1}} + \frac{1}{p-1} = \frac{1}{p-1}$.
- Thus $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$.

Suppose $0 < p < 1$.

- $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty} \frac{x^{1-p}}{1-p}\bigg|_0^b = \lim_{b\to\infty} \frac{b^{1-p}}{1-p} - \frac{1}{1-p} = \infty$ since $1-p > 0$.
- So $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ diverges for $0 < p < 1$.

9. Determine which of the following series converges conditionally, converges absolutely or diverges. Specify which convergence test you use and show how it leads to the answer.

(a) $\displaystyle\sum_{n=2}^\infty \frac{(-1)^n}{\ln n}$ Test for absolute convergence first.

Compare $\displaystyle\sum_{n=1}^\infty \left|(-1)^n \frac{1}{\ln n}\right| = \sum_{n=1}^\infty \frac{1}{\ln n}$ to $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ using the Comparison Test.

$\ln n < n \implies \dfrac{1}{\ln n} > \dfrac{1}{n}$ To show $\ln n < n$ for $n \geq 2$, we look at

- $f(x) = x - \ln x$. Then $f'(x) = 1 - 1/x > 0$ for $x > 1$, so $f(x)$ increasing for $x > 1$. Thus $x \geq 2 \implies f(x) \geq f(2) = 2 - \ln 2 > 0$, so $x > \ln x$ for $x \geq 2$.

$\displaystyle\sum_{n=2}^\infty \frac{1}{n}$ diverges $\implies \displaystyle\sum_{n=2}^\infty \frac{1}{\ln n}$ diverges (by the Comparison Test)

Now use the Alternating Series Test since the series is not absolutely convergent.

Set $b_n = 1/\ln n$. Two ways to show b_n is decreasing: (1) the denominators $\ln n$ are increasing or (2) consider $f(x) = 1/\ln x$ for $x > 1$ and its first derivative:

$$f'(x) = -\frac{1}{(\ln x)^2}\frac{1}{x} < 0.$$

That is negative, so $f(x)$ is decreasing for $x > 1$.

Since $b_n \to 0$ as $n \to \infty$, the series converges by the Alternating Series Test but is not absolutely convergent. It is conditionally convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+50}$

1. $\displaystyle\lim_{n\to\infty} \frac{n^2}{n^2+50} = \lim_{n\to\infty} \frac{1}{1+50/n^2} = 1 \neq 0$

2. The series diverges by the Divergence Test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^5}{n!}$ Apply the Ratio Test to test for absolute convergence.

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\dfrac{(n+1)^5}{(n+1)!}}{\dfrac{n^5}{n!}} = \frac{(n+1)^5}{n^5} \frac{n!}{(n+1)!} = \left(\frac{n+1}{n}\right)^5 \frac{1}{n+1} = \left(1+\frac{1}{n}\right)^5 \frac{1}{n+1}$$

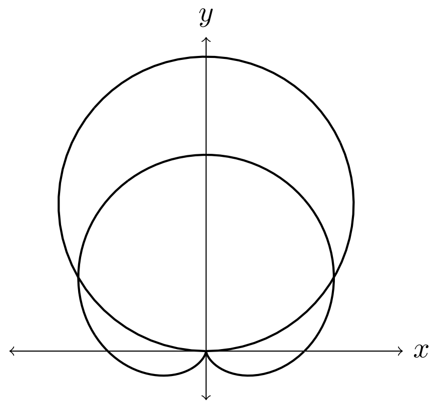
$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^5 \frac{1}{n+1} = 0 < 1.$$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^5}{n!}$ converges absolutely.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{5}{2^n+5n+3}$ Apply the Comparison Test (terms all positive, so no need for absolute values)

$0 < \dfrac{5}{2^n+5n+3} < \dfrac{5}{2^n}$ and $\displaystyle\sum_{n=0}^{\infty} \frac{5}{2^n}$ converges since this is a geometric series, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2^n+5n+3}$ converges absolutely using the Comparison Test. (Note: The Ratio Test and Limit Comparison Test could be used also.)

10. Below are graphs of $r = 3\sin\theta$ and $r = 1 + \sin\theta$.



(a) Determine both polar and rectangular coordinates for all points where the curves cross in the first and second quadrants (not including the origin).

Finding crossing points:

$$3\sin\theta = 1 + \sin\theta \iff 2\sin\theta = 1 \iff \sin\theta = \frac{1}{2}.$$

Thus $\theta = \pi/6$ and $5\pi/6$ are choices for the angle at crossing points, and here $r = 3\sin\theta = 3/2$.

Polar coordinates of crossing points: $(r,\theta) = (3/2, \pi/6),\ (3/2, 5\pi/6)$

Rectangular coordinates: $(x, y) = (r\cos\theta, r\sin\theta) = (3\sqrt{3}/4, 3/4),\ (-3\sqrt{3}/4, 3/4)$

(b) Set up, but do **not** evaluate, an integral for the area of the region inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$.

The rays $\pi/6 \le \theta \le 5\pi/6$ "sweep out" the region.

To see which curve corresponds to which equation, write $r = 3\sin\theta$ in rectangular coordinates: $r^2 = 3r\sin\theta$ is the same as $x^2 + y^2 = 3y$, so $x^2 + (y^2 - (3/2)y + 9/4) = 9/4$, so $x^2 + (y - 3/2)^2 = (3/2)^2$. That is a circle centered at $(0, 3/2)$ with radius 3. The other curve is $r = 1 + \sin\theta$.

The area inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$ is

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left((3\sin\theta)^2 - (1 + \sin\theta)^2\right)\, d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left((3\sin\theta)^2 - (1 + \sin\theta)^2\right)\, d\theta.$$

11. Solve for y exactly:

(a) $\dfrac{dy}{dx} = \dfrac{\sin x}{y^2}$ with $y(0) = 3$.

1. Write as $y^2\ dy = \sin x\ dx$
2. $\displaystyle\int y^2\ dy = \int \sin x\ dx$
3. $\dfrac{y^3}{3} = -\cos x + C$
4. Apply initial condition $y(0) = 3$: $9 = -1 + C \Longrightarrow C = 10$.
5. Thus $\dfrac{y^3}{3} = 10 - \cos x \Longrightarrow y = (30 - 3\cos x)^{1/3}$.

(b) $\dfrac{dy}{dx} = y\cos x + xy$ with $y(0) = 3$.

1. $\dfrac{dy}{dx} = y(\cos x + x) \Longrightarrow \dfrac{1}{y}\ dy = (\cos x + x)\,dx$
2. $\displaystyle\int \frac{1}{y}\ dy = \int (\cos x + x)\,dx$
3. $\ln|y| = \sin x + \dfrac{x^2}{2} + C \Longrightarrow |y| = e^{\sin x + x^2/2 + C} = e^C e^{\sin x + x^2/2}$
4. $y = \pm e^C e^{\sin x + x^2/2} = Ke^{\sin x + x^2/2}$, where $K = \pm e^C$.

Apply initial condition $y(0) = 3$: $3 = Ke^{\sin 0 + 0^2/2} = K$

Solution is $y = 3e^{\sin x + x^2/2}$.

12. Find the orthogonal trajectories of the family of curves $y = kx^4$, where k is an arbitrary (nonzero) constant.

For points (x, y) on the curve $y = kx^4$, we have $k = y/x^4$ and

$$\frac{dy}{dx} = 4kx^3 = 4\frac{y}{x^4}x^3 = \frac{4y}{x}.$$

A curve orthogonal to this family must have derivative (slope) equal to the negative reciprocal of $4y/x$, hence an orthogonal curve to the family has

$$\frac{dy}{dx} = \frac{-x}{4y}.$$

Now we solve this equation.

- $4y\,dy = -x\,dx$

- $\int 4y\,dy = -\int x\,dx$
- $2y^2 = -\frac{1}{2}x^2 + C$ for any C.

Thus the families $y = kx^4$ and $\frac{1}{2}x^2 + 2y^2 = C$ are **orthogonal trajectories** of each other.

13. A tank contains 30 L of water with 3 kg of salt dissolved in it. Brine that contains 5 kg of salt per liter enters the tank at a rate of 4 L/min. The solution in the tank is kept well mixed and is drained at a rate of 4 L/min. Use differential equations to determine how much salt remains in the tank after 30 minutes.

Let $y(t)$ = amount of salt (in kg) after t minutes, so $y(0) = 3$ kg.

Then dy/dt = inflow rate of salt − outflow rate of salt.

The volume remains constant since the rate of flow in is the same as the rate of the flow out, namely 4 L/min, so the concentration (in kg/L) of salt in the tank after t minutes is $y(t)/30$.

- inflow rate $= 5\frac{\text{kg}}{L} \times 4\frac{\text{L}}{\text{min}} = 20\frac{\text{kg}}{\text{min}}$.
- outflow rate $= \frac{y(t)}{30}\frac{\text{kg}}{L} \times 4\frac{\text{L}}{\text{min}} = \frac{4y(t)}{30}\frac{\text{kg}}{\text{min}}$

Our differential equation becomes $\frac{dy}{dt} = 20 - \frac{4y(t)}{30}$ with initial condition $y(0) = 3$.

$\frac{dy}{dt} = \frac{600 - 4y}{30} \Longrightarrow \frac{dy}{600 - 4y} = \frac{1}{30}\,dt$

- $\int \frac{dy}{600 - 4y} = \int \frac{1}{30}\,dt$
- $-\frac{1}{4}\ln|600 - 4y| = \frac{t}{30} + C \Longrightarrow \ln|600 - 4y| = -\frac{4t}{30} - 4C \Longrightarrow |600 - 4y| = e^{-4C}e^{-4t/30}$
- $600 - 4y = \pm e^{-4C}e^{-4t/30} = Ke^{-4t/30} \Longrightarrow y(t) = (600 - Ke^{-4t/30})/4 = 150 - (K/4)e^{-4t/30}$

From the initial condition $y(0) = 3$ we have $3 = 150 - K/4$, so $K/4 = 150 - 3 = 147$. Thus the solution to the differential equation is $y(t) = 150 - 147e^{-4t/30}$.

$y(30) = 150 - 147e^{-4(30)/30} \approx 147.3$ kg

14. Compute $\int_0^\infty e^{-2x}\sin(x)\,dx$. Show all work.

We integrate by parts, starting with $u = e^{-2x}$ and $dv = \sin x\,dx$. Then $du = -2e^{-2x}\,dx$ and $v = -\cos x$, so

$$\int e^{-2x}\sin x\,dx = \int u\,dv = uv - \int v\,du = -e^{-2x}\cos x - 2\int e^{-2x}\cos x\,dx.$$

Doing integration by parts again with $u = e^{-2x}$ and $dv = \cos x\,dx$, so $du = -2e^{-2x}\,dx$ and $v = \sin x$,

$$\int e^{-2x}\cos x\,dx = \int u\,dv = uv - \int v\,du = e^{-2x}\sin x + 2\int e^{-2x}\sin x\,dx,$$

so

$$\begin{aligned}\int e^{-2x}\sin x\,dx &= -e^{-2x}\cos x - 2\left(e^{-2x}\sin x + 2\int e^{-2x}\sin x\,dx\right)\\ &= -e^{-2x}\cos x - 2e^{-2x}\sin x - 4\int e^{-2x}\sin x\,dx,\end{aligned}$$

so $5\int e^{-2x}\sin x\,dx = -e^{-2x}\cos x - 2e^{-2x}\sin x$. Therefore

$$\int e^{-2x}\sin x\,dx = -\frac{e^{-2x}\cos x}{5} - \frac{2e^{-2x}\sin x}{5} = -\frac{\cos x}{5e^{2x}} - \frac{2\sin x}{5e^{2x}}.$$

Therefore

$$\begin{aligned}\int_0^\infty e^{-2x}\sin x\,dx &= \lim_{b\to\infty}\int_0^b e^{-2x}\sin x\,dx\\ &= \lim_{b\to\infty} - -\frac{\cos x}{5e^{2x}} - \frac{2\sin x}{5e^{2x}}\Bigg|_0^b\\ &= \lim_{b\to\infty} -\frac{\cos b}{5e^{2b}} - \frac{2\sin b}{5e^{2b}} + \frac{1}{5}\\ &= \frac{1}{5}\end{aligned}$$

since $|(\cos b)/e^{2b}| \le 1/e^{2b} \to 0$ and $|(\sin b)/e^{2b}| \le 1/e^{2b} \to 0$ as $b \to \infty$.

15. Compute $\displaystyle\int \frac{dx}{x^2 - ax}$, where $a \neq 0$. Your answer will depend on a. Show all work.

First we decompose $\dfrac{1}{x^2 - ax} = \dfrac{1}{x(x-a)}$ into partial fractions:

$$\frac{1}{x(x-a)} = \frac{A}{x} + \frac{B}{x-a}.$$

Multiply both sides by $x(x-a)$ to get

$$1 = A(x-a) + Bx.$$

Set $x = a$ and $x = 0$ to get $1 = aB$ and $1 = -aA$, so $A = -1/a$ and $B = 1/a$. Thus

$$\frac{1}{x(x-a)} = -\frac{1}{ax} + \frac{1}{a(x-a)}.$$

Integrating both sides,

$$\int \frac{dx}{x(x-a)} = -\frac{1}{a}\int\frac{dx}{x} + \frac{1}{a}\int\frac{dx}{x-a} = -\frac{1}{a}\ln|x| + \frac{1}{a}\ln|x-a| + C.$$

Error Bound Formulas

Trapezoid Rule and Error Bound: Let $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ with $x_i - x_{i-1} = \Delta x = \dfrac{b-a}{n}$ for all i. The nth approximation T_n to $\displaystyle\int_a^b f(x)\,dx$ using the Trapezoid rule is

$$T_n = (f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n))\frac{\Delta x}{2}$$

and the error in approximating the integral by T_n can be bounded as:

$$\left|\int_a^b f(x)\,dx - T_n\right| \leq \frac{K(b-a)}{12}(\Delta x)^2 = \frac{K(b-a)^3}{12n^2},$$

where K is any upper bound on $|f''(x)|$ over $[a,b]$: $|f''(x)| \leq K$ for $a \leq x \leq b$.

Taylor's Inequality: Let $T_n(x)$ be the nth-order Taylor polynomial for $f(x)$ at $x = a$. If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$ then

$$|T_n(x) - f(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!} \quad \text{for} \quad |x-a| \leq d.$$