Integration Practice Exam Sol^{ns}

1. $\int \sin^2 x \, dx$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\int \frac{(1-\cos 2x)}{2} dx$

$= \frac{1}{2}\int 1 \, dx - \frac{1}{2}\int \cos 2x \, dx$

$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + c$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

2. $\int \sin^3 x \, dx$

$\int \sin^2 x \cdot \sin x \, dx$

$\int (1-\cos^2 x) \cdot \sin x \, dx$

$u = \cos x \quad du = -\sin x \, dx$

$\int (1-u^2)(-du) = \int (u^2-1) \, du$

$= \frac{u^3}{3} - u + c$

$\frac{\cos^3 x}{3} - \cos x + c$

3. $\int \sin x \cdot \cos x \, dx$

$u = \sin x \quad du = \cos x \, dx$

$\int u \, du = \frac{u^2}{2} + c$

$= \frac{\sin^2 x}{2} + c$

For $u = \cos x$
You get $-\frac{1}{2}\cos^2 x + c$ same answer upto '+c'

4. $\int \frac{\tan x}{\cos^2 x} dx$

$= \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos^2 x} dx$

$u = \cos x \quad du = -\sin x \, dx$

$\int \frac{-du}{u^3} = -\int u^{-3} du$

$= -\frac{u^{-2}}{-2} + c = \frac{1}{2}\frac{1}{\cos^2 x} + c$

$= \frac{1}{2}\sec^2(x) + c$

(5) $\int \sin x \cdot \cos^3 x \, dx$

$u = \cos x \quad du = -\sin x \, dx$

$\int -u^3 \cdot du$

$= -\int u^3 du = -\frac{u^4}{4} + c$

$= -\frac{\cos^4 x}{4} + c$

(6) $\int x^2 e^x \, dx$

$dv = e^x dx \quad u = x^2$

$v = e^x \quad du = 2x\,dx$

$\int x^2 e^x dx = x^2 \cdot e^x - \int e^x \cdot 2x\,dx$

$= x^2 \cdot e^x - 2\int x \cdot e^x dx$

$u = x \quad dv = e^x dx$

(Integration by parts 2nd time)

$du = dx \quad v = e^x$

$= x^2 e^x - 2\left[x \cdot e^x - \int e^x \cdot dx\right]$

$= x^2 \cdot e^x - 2xe^x + 2e^x + c$

(7) $\int x \cos(2x) \, dx$

$u = x \quad dv = \cos(2x) dx$

$du = dx \quad v = \frac{\sin(2x)}{2}$

$\int x \cos(2x) dx = x \cdot \frac{\sin(2x)}{2} - \int \frac{\sin(2x)}{2} \cdot dx$

$= x \frac{\sin(2x)}{2} - \frac{1}{2}\left(-\frac{\cos(2x)}{2}\right) + c$

$= \frac{x \sin(2x)}{2} + \frac{1}{4}\cos(2x) + c$

(8) $\int x^2 \cos x \, dx$

$u = x^2 \quad dv = \cos x \, dx$

$du = 2x\,dx \quad v = \sin x$

$\int x^2 \cos x \, dx = x^2 \sin x - \int \sin x \cdot 2x\,dx$

$= x^2 \sin x - 2\int x \sin x \, dx$

Integration by parts 2nd time

$= x^2 \sin x - 2\int x \sin x \, dx$

$u = x \quad dv = \sin x \, dx$

$du = dx \quad v = -\cos x$

$= x^2 \sin x - 2\left[x \cdot (-\cos x) - \int -\cos x \, dx\right]$

$= x^2 \sin x + 2x\cos x - 2\sin x + c$

9) $\int x^3 \ln x \, dx$

$u = \ln x \qquad dv = x^3 dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^4}{4}$

$\int x^3 \ln(x) \, dx = \ln x \cdot \frac{x^4}{4} - \int \frac{1}{x} dx \cdot \frac{x^4}{4}$

$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 dx$

$= \frac{x^4}{4} \ln x - \frac{1}{16} \cdot x^4 + c$

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10) $\int \frac{dx}{x^2 - 9} = \int \frac{dx}{(x-3)(x+3)}$

$\frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$

$1 = A(x+3) + B(x-3)$

$= Ax + Bx + 3A - 3B$

$= \cancel{A+x}(A+B)x + 3A - 3B$

$A + B = 0$

$3A - 3B = 1$

$6A = 1 \quad A = 1/6 \quad B = -1/6$

$\int \frac{dx}{(x-3)(x+3)} = \int \frac{1/6}{(x-3)} dx + \int \frac{-1/6}{(x+3)} dx$

$= \frac{1}{6} \ln|x-3| - \frac{1}{6} \ln|x+3| + c$

11) $\int \frac{6x+2}{x^2+3x+2} dx$

$\frac{6x+2}{x^2+3x+2} = \frac{6x+2}{(x+2)(x+1)}$

$\frac{6x+2}{(x+2)(x+1)} = \frac{A}{(x+2)} + \frac{B}{(x+1)}$

$6x + 2 = A(x+1) + B(x+2)$

$6x + 2 = Ax + Bx + A + 2B$

$6x + 2 = (A+B)x + A + 2B$

$A + B = 6$

$A + 2B = 2$

$-B = 4 \quad B = -4 \quad A = 10$

$\int \frac{6x+2}{(x+2)(x+1)} dx = \int \frac{10}{(x+2)} dx + \int \frac{-4}{(x+1)} dx$

$= 10 \ln|x+2| - 4 \ln|x+1| + c$

(12) $\displaystyle\int \frac{x+2}{x^2+3x-4}\,dx$

$$\frac{x+2}{x^2+3x-4} = \frac{x+2}{(x-1)(x+4)} = \frac{A}{(x-1)} + \frac{B}{(x+4)}$$

$$x+2 = A(x+4)+B(x-1)$$

$$x+2 = (A+B)x + 4A - B$$

$$A+B = 1$$
$$4A - B = 2$$
$$5A = 3 \quad A = 3/5$$
$$B = 1-3/5 = 2/5$$

$$\int \frac{x+2}{x^2+3x-4}\,dx = \int \frac{A}{(x-1)}\,dx + \int \frac{B}{x+4}\,dx$$

$$= \int \frac{3/5}{(x-1)}\,dx + \int \frac{2/5}{(x+4)}\,dx$$

$$= \frac{3}{5}\ln|x-1| + \frac{2}{5}\ln|x+4| + C$$

(13) $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{(x-1)} + \frac{Bx+C}{x^2+9}$$

$$10 = A(x^2+9)+(Bx+C)(x-1)$$
$$= Ax^2+Bx^2+9A-Bx+Cx-C$$
$$= (A+B)x^2-(B-C)x+9A-C$$

$$A+B = 0$$
$$B-C = 0 \quad B=C$$
$$9A-C = 10$$

$$A+C = 0$$
$$9A-C = 10$$
$$10A = 10$$
$$A = 1$$
$$B = -1$$
$$C = -1$$

$$\int \frac{10}{(x-1)(x^2+9)}\,dx = \int \frac{A}{(x-1)}\,dx + \int \frac{Bx+C}{x^2+9}\,dx$$

$$= \int \frac{1}{x-1}\,dx + \int \frac{-x-1}{x^2+9}\,dx$$

$$= \ln|x-1| + \int \frac{-x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$$

$$= \ln|x-1| - \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

(14) $\displaystyle\int \frac{4x}{(x+1)(x^2+1)}\,dx$

$$\frac{4x}{(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+1)}$$

$$4x = A(x^2+1) + (Bx+c)(x+1)$$

$$4x = Ax^2 + A + Bx^2 + Bx + Cx + C$$

$$4x = (A+B)x^2 + (B+c)x + A+C$$

$A+B=0 \quad B=-A$

$B+C=4 \rightarrow -A-A=4 \quad -2A=4$

$A+C=0 \quad C=-A \quad A=-2$

$B=2,\ c=2$

$$\int \frac{4x}{(x+1)(x^2+1)}\,dx = \int \frac{-2}{(x+1)}\,dx + \int \frac{2x+2}{(x^2+1)}\,dx$$

$$\boxed{= -2\ln|x+1| + \ln|x^2+1| + 2\tan^{-1}(x) + c}$$

Note: $\displaystyle\int \frac{2x\,dx}{x^2+1} + \int \frac{2}{x^2+1}\,dx = ?$

$u = x^2+1$

$du = 2x\,dx$

$\displaystyle\int \frac{1}{u}\,du$

$\ln|u| + c$

$\ln|x^2+1| + c$

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$\displaystyle 2\int \frac{\sec^2\theta\,d\theta}{\sec^2\theta} = 2\int 1\,d\theta$

$= 2\cdot\theta + c$

$= 2\tan^{-1}(x) + c$

(15) $\displaystyle\int \frac{t+3}{t^3-t}\,dt$

$$\frac{t+3}{t^3-t} = \frac{t+3}{t(t^2-1)} = \frac{t+3}{t(t-1)(t+1)}$$

$$\frac{t+3}{t(t-1)(t+1)} = \frac{A}{t} + \frac{B}{(t-1)} + \frac{C}{t+1}$$

$$t+3 = A(t^2-1) + B\cdot t(t+1) + C\cdot t(t-1)$$

$$t+3 = At^2 - A + Bt^2 + Bt + ct^2 - ct$$

$$= (A+B+c)t^2 + (B-c)t - A$$

$-A = 3 \quad A = -3$

$B - C = 1 \qquad B - c = 1$

$A+B+C = 0 \qquad B + C = 3$

$2B = 4 \Rightarrow B = 2$

$C = 1$

$$\int \frac{t+3}{t^3-1}\,dt = \int \frac{A}{t}\,dt + \int \frac{B}{t-1}\,dt + \int \frac{c}{t+1}\,dt$$

$$= \int \frac{-3}{t}\,dt + \int \frac{2}{t-1}\,dt + \int \frac{1}{t+1}\,dt$$

$$-3\ln|t| + 2\ln|t-1| + \ln|t+1| + C$$

(16) $$\int \frac{2x-3}{x^3+4x}\,dx$$

$$\frac{2x-3}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$2x-3 = A(x^2+4)+(Bx+C)x$$

$$2x-3 = Ax^2+4A+Bx^2+Cx$$

$$2x-3 = (A+B)x^2+cx+4A$$

$C=2$ $\qquad$ $A+B=0$

$4A=-3 \quad A=-3/4$ $\qquad$ $B=-A$

$B=-(-3/4)$

$A=-3/4;\ B=3/4;\ C=2$

$$\int \frac{-3/4}{x}\,dx + \int \frac{\frac{3}{4}x+2}{x^2+4}\,dx$$

$$= \frac{-3}{4}\int \frac{1}{x}\,dx + \frac{3}{4}\int \frac{x}{x^2+4}\,dx + \int \frac{2}{x^2+4}\,dx$$

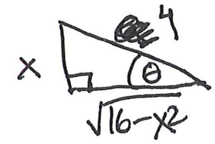
$$= \frac{-3}{4}\ln|x| + \frac{3}{4}\cdot\frac{1}{2}\ln|x^2+4| + \tan^{-1}\left(\frac{x}{2}\right)$$

$+c$

(17) $$\int \frac{dx}{\sqrt{16-x^2}}$$

this is of the type $\sqrt{a^2-x^2}$

$x = a\sin\theta.$



$\therefore\ x = 4\sin\theta.$

$dx = 4\cos\theta\,d\theta.$

$$\sqrt{16-16\sin^2\theta} = \sqrt{16\cos^2\theta}$$
$$= 4\cos\theta.$$

$$\int \frac{dx}{\sqrt{16-x^2}} = \int \frac{4\cdot\cos\theta\,d\theta}{4\cdot\cos\theta} = \int 1\,d\theta$$

$= \theta + c$

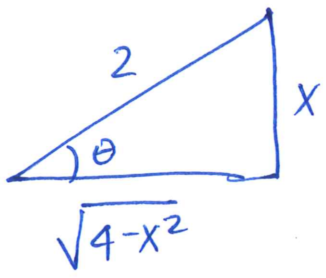
$x = 4\sin\theta \Rightarrow \sin\theta = \frac{x}{4}$

$\theta = \sin^{-1}\left(\frac{x}{4}\right)$

$$\therefore \int \frac{dx}{\sqrt{16-x^2}} = \sin^{-1}\left(\frac{x}{4}\right) + c$$

18. $\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$.

$x = 2\sin\theta \qquad \sin\theta = \frac{x}{2}$



$x^2\sqrt{4-x^2} = 4\sin^2\theta\sqrt{4-4\sin^2\theta}$

$= 4\sin^2\theta \cdot 2\cos\theta$

$dx = 2\cos\theta\, d\theta$

$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{\cancel{2}\cancel{\cos\theta}\, d\theta}{4\sin^2\theta\,\cancel{2}\cancel{\cos\theta}}$

$= \frac{1}{4}\int \csc^2\theta = -\frac{\cot\theta}{4} + c$

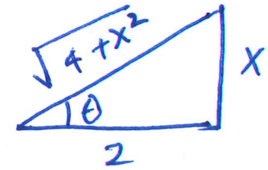
From the triangle $\cot\theta = \frac{1}{\tan\theta}$

$= \frac{\sqrt{4-x^2}}{x}$

$\int \frac{dx}{x^2\sqrt{4-x^2}} = \frac{-\sqrt{4-x^2}}{4x} + C$

19. $\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

$x = 2\tan\theta \qquad \frac{x}{2} = \tan\theta$



$dx = 2\sec^2\theta\, d\theta$

$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{8\tan^3\theta \cdot \cancel{2}\cdot\sec^{\cancel{2}}\theta\, d\theta}{\cancel{2}\,\cancel{\sec\theta}}$

$= \int 8\tan^3\theta \cdot \sec\theta\, d\theta$

$= \int 8(\tan^2\theta)\cdot\sec\theta\cdot\tan\theta\, d\theta$

$= 8\int(\sec^2\theta - 1)\cdot\sec\theta\tan\theta\, d\theta$

$u = \sec\theta \quad . \quad du = \sec\theta\tan\theta\, d\theta$

$= 8\int(u^2-1)\,du$

$= 8\cdot\frac{u^3}{3} - 8\cdot u + c$

$= 8\cdot\frac{\sec^3\theta}{3} - 8\sec\theta + c$

From the triangle $\sec\theta = \frac{1}{\cos\theta}$

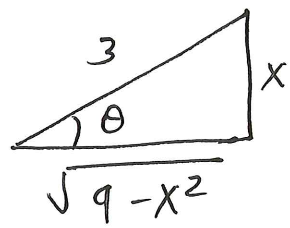
$= \frac{\sqrt{4+x^2}}{2}$

$= \frac{8}{3}\,\frac{(4+x^2)^{3/2}}{8} - 8\cdot\frac{(4+x^2)^{1/2}}{2} + c$

(20). $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$ $\quad x = 3\sin\theta \quad \sin\theta = \frac{x}{3}$

$dx = 3\cos\theta\, d\theta$

$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\cdot\cos\theta$



$$\int \frac{x^2\cdot dx}{\sqrt{9-x^2}} = \int \frac{9\cdot \sin^2\theta \cdot \cancel{3}\cancel{\cos\theta}\, d\theta}{\cancel{3}\cdot\cancel{\cos\theta}} = \int 9\sin^2\theta\, d\theta$$

$$= 9\int \frac{1-\cos 2\theta}{2}\, d\theta = \frac{9}{2}\int 1\cdot d\theta - \frac{9}{2}\int \cos 2\theta\, d\theta$$

$$= \frac{9}{2}\theta - \frac{9}{2}\frac{\sin 2\theta}{2} = \frac{9}{2}\theta - \frac{9}{4}\cdot 2\sin\theta\cdot\cos\theta + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\cdot\sqrt{9-x^2} + C$$