9.3 Separable Equations

Separable equation. A first-order differential equation is called *separable* if it has the form $\frac{dy}{dx} = g(x)f(y)$. The right side is a product of separate functions of x and of y.

Orthogonal trajectory. An *orthogonal trajectory* of a family of curves is a curve intersecting each curve of the family at right angles (orthogonally). Since a line perpendicular to a line with slope m has slope $-\frac{1}{m}$, if a family of curves has derivative $\frac{dy}{dx} = m(x, y)$, then the derivative of an orthogonal trajectory to that family of curves is $\frac{dy}{dx} = -\frac{1}{m(x, y)}$.

Mixing problems. If $y(t)$ denotes the amount of substance in a tank at time t, then its rate of change is $\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$.

Example: A tank contains 500 L of brine with 15 kg of dissolved salt. Brine having .2 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and is drained from the tank at 10 L/min. How much salt is in the tank after t minutes? After 20 min?

Thinking about the problem:

Let $y(t)$ be the amount of salt in the tank at t min. We need the rate in and rate out of salt in kg/min. The rate in is the concentration of salt (in kg/L) multiplied by the rate of liquid entering the tank (in L/min), and the rate out is the concentration of salt multiplied by the rate of liquid leaving the tank. After finding $\frac{dy}{dt}$, we solve for $y(t)$ and $y(20)$.

Doing the problem:

Let $y(t)$ be the amount of kg of salt in the tank at t minutes, so $y(0) = 15$: reread the first sentence in the problem. The problem also says brine with .2 kg/L of salt enters at a rate of 10 L/min and the whole mixture drains from the tank at 10 L/min. The concentration of salt entering the tank at time t is .2 kg/L, so the rate of salt entering the tank at time t is

$$\text{concentration} \cdot \text{rate of liquid entering the tank} = .2\,\frac{\text{kg}}{\text{L}} \cdot 10\,\frac{\text{L}}{\text{min}} = 2\,\frac{\text{kg}}{\text{min}}.$$

The concentration of salt leaving the tank at time t is

$$\frac{\text{amount of salt in tank}}{\text{volume of tank}} = \frac{y(t)\,\text{kg}}{500\,\text{L}},$$

so the rate of salt leaving of the tank at time t is

$$\text{concentration} \cdot \text{rate of liquid leaving the tank} = \frac{y(t)\text{kg}}{500\,\text{L}} \cdot 10\,\frac{\text{L}}{\text{min}} = \frac{y(t)}{50}\,\frac{\text{kg}}{\text{min}}.$$

Therefore, in kg/min,

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out}) = 2 - \frac{y(t)}{50} = \frac{100 - y(t)}{50}$$

The differential equation $\boxed{\frac{dy}{dt} = \frac{100 - y}{50}}$ is separable:

$$\frac{dy}{dt} = \frac{100 - y}{50} \Longrightarrow \frac{dy}{100 - y} = \frac{dt}{50} \Longrightarrow \int \frac{dy}{100 - y} = \int \frac{dt}{50} \Longrightarrow -\ln|100 - y| = \frac{t}{50} + C.$$

Thus $\ln|100 - y| = -t/50 - C$, so raising e to both sides, we get $100 - y(t) = \pm e^{-C} e^{-t/50}$. Setting $t = 0$ here, $100 - 15 = \pm e^{-C}$, so $100 - y(t) = 85e^{-t/50}$. Thus $y(t) = 100 - 85e^{-t/50}$. This is the number of kilograms of salt in the tank after t minutes. After 20 minutes there is $y(20) = 100 - 85e^{-20/50} \approx 43.02$ kg of salt.

Remark. In the long run, the concentration of salt in the tank must match that of incoming brine (.2 kg/L), so the amount of salt in 500 L should tend to (.2 kg/L)(500 L)= 100 kg, which is consistent with $y(t) \to 100$ as $t \to \infty$.

Solutions should show all of your work, not just a single final answer.

1. Find the solution of $\frac{dy}{dx} = e^x e^y$ where $y(0) = 1$.

2. Find the general solution of the differential equation $(y^2 + xy^2)y' = 1$.

3. Find the orthogonal trajectories of the family of curves $y^4 = kx^3$, where k is constant.

4. A tank contains 1000 L of pure water. Brine that contains .5 kg of salt per liter of water enters the tank at a rate of 20 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 20 L/min. How many kg of salt are in the tank after t minutes? After one hour (round to one digit after the decimal point)?

5. T/F (with justification)

The differential equation $\frac{dy}{dx} = yx + y$ is separable.