8.1 Arc Length

The Arc Length Formula. If $f'(x)$ is continuous for x in $[a,b]$, then the length of the curve $y = f(x)$ for $a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

If $g'(y)$ is continuous for y in $[c,d]$, then the length of the curve $x = g(y)$ for $c \le y \le d$, is

$$L = \int_c^d \sqrt{1 + g'(y)^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Example: Write the arc length of $y = \sqrt{x}$ for $1 \le x \le 2$ as a definite integral with respect to x.

Thinking about the problem:

The arc length formula to use is $L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$, so we need to find $f'(x)$ and check it is continuous on $[1,2]$.

Doing the problem:

The problem asks for the definite integral for the arc length of $y = \sqrt{x}$ on $1 \le x \le 2$. We use $f(x) = \sqrt{x} = x^{1/2}$, so $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$, which is continuous on $[1,2]$: the arc length is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx = \int_1^2 \sqrt{1 + \left(\frac{1}{2x^{1/2}}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{4x}}\, dx.$$

Solutions should show all of your work, not just a single final answer.

1. Write the arc length of $x = y^2$ for $1 \le x \le 2$ as a definite integral with respect to y.

2. (a) Write the arc length of $y = x^3$ for $0 \le x \le 2$ as a definite integral with respect to x.

   (b) Write the arc length of $y = x^3$ for $0 \le x \le 2$ as a definite integral with respect to y.

   (c) Why is the integral in part (b) improper?

3. T/F (with justification): The arc length of the curve $y = \sin x$ for $0 \le x \le \pi/2$ equals $\displaystyle\int_0^{\pi/2} \sqrt{1 + \sin^2 x}\, dx$.