7.8 Improper Integrals

A Riemann integral uses bounded (continuous) functions on bounded intervals. An improper integral has one of the conditions break down: an infinite domain of integration or an unbounded function, which usually means a vertical asymptote.

When the domain is infinite, either $[a, \infty)$ and $(-\infty, b]$, we set

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx, \qquad \int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

if the limit exists, and

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

where we pick c anywhere that makes the integrals on the right meaningful.

When the function is defined on $[a, b]$ except for a vertical asymptote at an endpoint, set

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx, \qquad \int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

when there is a vertical asymptote at b in the first integral and a vertical asymptote at a in the second integral. If instead there is a vertical asymptote at a number c in (a, b), and the improper integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ make sense, then we set

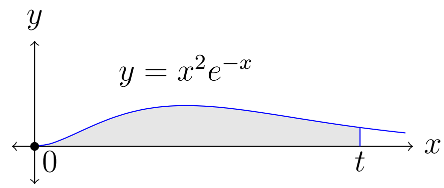
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

When an improper integral makes sense we call it *convergent*. Otherwise it's called *divergent*.

Example: Is $\int_0^\infty x^2e^{-x}\,dx$ convergent or divergent? If convergent, evaluate it.

Thinking about the problem:

The integral is $\lim\limits_{t\to\infty} \int_0^t x^2e^{-x}\,dx$ and the graph of $y = x^2e^{-x}$ is below. We will compute $\int_0^t x^2e^{-x}\,dx$ and see how it behaves as $t \to \infty$.



To evaluate $\int_0^t x^2 e^{-x}\,dx$ we will use integration by parts. We did this for $\int x^2 e^x\,dx$ on Worksheet 7.1, and the calculations will be similar.

Doing the problem:

To work out $\int_0^t x^2 e^{-x}\,dx$ with integration by parts set u and dv to be as in the chart below, and then compute du and v.

$u = x^2$	$dv = e^{-x}\,dx$
$du = 2x\,dx$	$v = -e^{-x}$

Thus $\int_0^t x^2 e^{-x}\,dx = uv\Big|_0^t - \int_0^t v\,du = -x^2 e^{-x}\Big|_0^t + \int_0^t 2xe^{-x}\,dx = -\frac{t^2}{e^t} + 2\int_0^t xe^{-x}\,dx$. We work out the new integral also using integration by parts, starting with the chart below.

$u = x$	$dv = e^{-x}\,dx$
$du = dx$	$v = -e^{-x}$

Thus

$$\int_0^t xe^{-x}\,dx = -xe^{-x}\Big|_0^t + \int_0^t e^{-x}\,dx = -\frac{t}{e^t} - e^{-x}\Big|_0^t = -\frac{t}{e^t} - \frac{1}{e^t} + 1,$$

so returning to the initial calculation we have

$$\int_0^t x^2 e^{-x}\,dx = -\frac{t^2}{e^t} + 2\int_0^t xe^{-x}\,dx = -\frac{t^2}{e^t} + 2\left(-\frac{t}{e^t} - \frac{1}{e^t} + 1\right) = -\frac{t^2}{e^t} - \frac{2t}{e^t} - \frac{2}{e^t} + 2.$$

Letting $t \to \infty$, $\int_0^\infty x^2 e^{-x}\,dx = \lim_{t\to\infty}\left(-\frac{t^2}{e^t} - \frac{2t}{e^t} - \frac{2}{e^t} + 2\right) = 0 - 0 - 0 + 2$ by L'Hospital's rule (used twice for the first expression). Thus $\int_0^\infty x^2 e^{-x}\,dx = 2$: the improper integral is convergent and equals 2.

Solutions should show all of your work, not just a single final answer.

1. Decide if $\int_0^{\infty} \frac{x}{x^2+1}\,dx$ is convergent or divergent.

   (a) Draw the graph of the function $\frac{x}{x^2+1}$ for $x \geq 0$ and shade the region that is to be integrated.

   (b) Evaluate the improper integral to determine if it is convergent or divergent. If it is convergent, specify its value.

2. We will evaluate $\int_0^\infty \frac{1}{e^x+1}dx$. (This improper integral occurs in quantum mechanics.)

(a) Draw the graph of the function $\frac{1}{e^x+1}$ for $x \geq 0$ and shade the region that is to be integrated.

(b) Use the Comparison Theorem to show that $\int_0^\infty \frac{1}{e^x+1}dx$ converges.

(c) Use the substitution $u = e^x + 1$ to turn $\int_0^\infty \frac{1}{e^x+1}\,dx$ into the improper integral of a rational function of u.

(d) Compute the improper integral of the rational function in part (b) as a limit of integrals with an upper bound t, as $t \to \infty$. (One of your WebAssign exercises for 7.8 is similar to this task.)

3. Determine if $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)}$ is convergent or divergent.

4. T/F (with justification) The integral $\displaystyle\int_0^2 \frac{dx}{x-1}$ is convergent.