7.4 Integration by Partial Fractions

Remember:

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \text{ if } a \neq b,$$

$$\frac{1}{x(x^2+a)} = \frac{A}{x} + \frac{Bx+C}{x^2+a} \text{ if } a \neq 0,$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \text{ if } a \neq 0.$$

Example. Evaluate $\int \frac{2x+1}{x^2-4}\,dx$.

Thinking about the problem:

The integrand $\frac{2x+1}{x^2-4}$ is a rational function and does not look like it can be handled with substitution, so we use partial fractions. The denominator x^2-4 is $(x+2)(x-2)$, a product of different linear factors, so the partial fraction decomposition of $\frac{2x+1}{x^2-4}$ is $\frac{A}{x+2} + \frac{B}{x-2}$ for some constants A and B. After solving for A and B we would have $\int \frac{2x+1}{x^2-4}\,dx = \int \frac{A}{x+2}\,dx + \int \frac{B}{x-2}\,dx$ and can integrate the right side.

Doing the Problem:

Writing $\frac{2x+1}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2}$, solve for A and B by multiplying both sides by the denominator x^2-4:

$$2x+1 = A(x-2) + B(x+2)$$

. Setting $x=2$, we find

$$2(2)+1 = 5 = A(0) + B(2+2) = 4B \Rightarrow 5 = 4B \Rightarrow B = \frac{5}{4}.$$

Setting $x = -2$, we find

$$2(-2) + 1 = A(-2 - 2) + B(0) \Rightarrow -3 = -4A \Rightarrow A = \frac{3}{4}.$$

Therefore $\dfrac{2x+1}{x^2-4} = \dfrac{3/4}{x+2} + \dfrac{5/4}{x-2}$, so

$$\begin{aligned}\int \frac{2x+1}{x^2-4}\,dx &= \frac{3}{4}\int \frac{dx}{x+2} + \frac{5}{4}\int \frac{dx}{x-2} \\ &= \frac{3}{4}\ln|x+2| + \frac{5}{4}\ln|x-2| + C.\end{aligned}$$

Solutions should show all of your work, not just a single final answer.

1. Evaluate $\displaystyle\int \frac{dx}{x^2 - 4x + 3}$.

   (a) Decompose the integrand into partial fractions.

   (b) Evaluate the integral using partial fractions.

2. Evaluate $\displaystyle\int \frac{x^2 + x + 1}{x(x^2 + 9)}\, dx.$

(a) Decompose the integrand into partial fractions.

(b) Evaluate the integral using partial fractions.

3. Evaluate $\displaystyle\int \frac{x+1}{x^3-8x^2+20x}\,dx$.

4. T/F (with justification): Computing $\displaystyle\int \frac{x}{x^2-1}\,dx$ requires partial fractions.