11.9 Representations of Functions as Power Series

Power Series, Derivatives, and Integrals. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

for x such that $|x-a| < R$ is differentiable (and therefore continuous) and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$,

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$.

The radii of convergence of the power series in (i) and (ii) are both R, although the interval of convergence of these series might not match the interval of convergence of $f(x)$.

Power Series for a Geometric Series: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ if $|x| < 1$.

Example: Find a power series centered at $x = 0$ for the function $\frac{1}{2-5x}$ and find its interval of convergence.

Thinking about the problem:

The function $f(x) = \frac{1}{2-5x}$ looks similar to $\frac{1}{1-x}$, so we will alter $f(x)$ to make it more closely resemble that. Factor 2 from the whole denominator: $\frac{1}{2-5x} = \frac{1}{2} \cdot \frac{1}{1-5x/2}$. We will write $\dfrac{1}{1-5x/2}$ as a geometric series by replacing x in $\dfrac{1}{1-x}$ with $5x/2$. The interval of convergence of the power series for $\dfrac{1}{1-x}$ is $(-1,1)$, and we will use this to find the interval of convergence of the power series for $\dfrac{1}{1-5x/2}$, which will give us the interval of convergence for the power series of $f(x)$ centered at $x=0$.

Doing the problem:

The problem is to find a power series of $f(x) = \dfrac{1}{2-5x}$ centered at $x=0$. Write

$$f(x) = \frac{1}{2-5x} = \frac{1}{2(1-5x/2)} = \frac{1}{2} \cdot \frac{1}{1-5x/2}.$$

In the power series representation $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$, replace x with $\dfrac{5x}{2}$:

$$\frac{1}{1-5x/2} = \sum_{n=0}^{\infty} \left(\frac{5x}{2}\right)^n \quad \text{for} \quad \left|\frac{5x}{2}\right| < 1.$$

Thus

$$f(x) = \frac{1}{2-5x} = \frac{1}{2} \cdot \frac{1}{1-5x/2} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{5x}{2}\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{5^n}{2^{n+1}} x^n \text{ for } \left|\frac{5x}{2}\right| < 1}.$$

We have found a power series for $f(x)$ centered at $x=0$ and that it converges precisely when $\left|\dfrac{5x}{2}\right| < 1$, which is the same as $|x| < \dfrac{2}{5}$. Therefore a power series for $f(x)$ centered at $x=0$ has interval of convergence $\boxed{\left(-\frac{2}{5}, \frac{2}{5}\right)}$.

Solutions should show all of your work, not just a single final answer.

1. Find a power series centered at $x = 0$ for $\dfrac{1}{1+x^4}$ and its interval of convergence. (Hint: what can you substitute for x in $\dfrac{1}{1-x}$ to turn it into $\dfrac{1}{1+x^4}$?)

2. We will find a power series centered at $x = 0$ for $\frac{1}{(1-x)^3}$ and its interval of convergence.

(a) What are the first and second derivatives of $\frac{1}{1-x}$?

(b) What are the power series of the first and second derivatives of $\frac{1}{1-x}$ centered at $x = 0$?

(c) What are the radii of convergence of the power series in (b)?

(d) Use (a), (b), and (c) to find a power series for $\frac{1}{(1-x)^3}$ centered at $x = 0$ and its interval of convergence.

3. Use power series to estimate $\displaystyle\int_0^{1/2} \frac{dx}{1+x^4}$ to within .00001 by the following steps.

(a) Express $\displaystyle\int \frac{dx}{1+x^4}$ as a power series centered at $x=0$, starting with the power series you found in problem 1.

(b) Find the radius of convergence of the power series in (a).

(c) Use (a) and (b) and the Alternating Series Estimation Theorem (Section 11.5) to estimate $\displaystyle\int_0^{1/2} \frac{dx}{1+x^4}$ to within .00001. Round your *estimate* to 5 digits after the decimal point.

4. T/F (with justification)

If $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 3 then $\displaystyle\sum_{n=0}^{\infty} c_n x^{2n}$ has radius of convergence 9.