11.6 Absolute Convergence and the Ratio Test

Absolute Convergence. A series $\sum a_n$ is called *absolutely convergent* if the series $\sum |a_n|$ with terms replaced by their absolute values is convergent.

A series that is absolutely convergent is convergent, but maybe not the other way around.

Conditional Convergence. A series $\sum a_n$ is called *conditionally convergent* if it is convergent but not absolutely convergent.

The Ratio Test.

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ then no conclusion can be drawn about convergence or divergence of $\sum_{n=1}^{\infty} a_n$: some examples converge and some examples diverge.

All parts of the ratio test apply to series starting at any n, not just starting at $n = 1$.

Series involving factorials often have their convergence determined using the ratio test, so it's important to be able to simplify ratios of factorials. For example,

$$\frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1}, \quad \frac{(2n+2)!}{(2n)!} = \frac{(2n+2)(2n+1)(2n)!}{(2n)!} = (2n+2)(2n+1).$$

Example: Determine if $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}$ is absolutely convergent, conditionally convergent, or divergent. Specify clearly which convergence test(s) you use.

Thinking about the problem:

Set $a_n = \frac{(2n)!}{(n!)^2}$. We have $a_n > 0$ for all n, so absolute convergence and convergence in this case mean the same thing. Since a_n has factorials in it, we will use the Ratio Test. We will simplify the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ and then compute its limit as $n \to \infty$.

Doing the problem:

We will use the Ratio Test. Let $a_n = \frac{(2n)!}{(n!)^2}$, so $a_{n+1} = \frac{(2(n+1))!}{((n+1)!)^2} = \frac{(2n+2)!}{((n+1)!)^2}$ and

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{a_{n+1}}{a_n} \\
&= \frac{\frac{(2n+2)!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}} \\
&= \frac{(2n+2)!(n!)^2}{((n+1)!)^2(2n)!} \\
&= \frac{(2n+2)!(n!)(n!)}{(n+1)!(n+1)!(2n)!} \\
&= \frac{(n!)(n!)(2n+2)!}{(n+1)!(n+1)!(2n)!} \\
&= \frac{n!}{(n+1)!} \cdot \frac{n!}{(n+1)!} \cdot \frac{(2n+2)!}{(2n)!} \\
&= \frac{1}{n+1} \cdot \frac{1}{n+1} \cdot (2n+2)(2n+1) \\
&= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \\
&= \frac{4n^2+6n+2}{n^2+2n+1}.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{4n^2+6n+2}{n^2+2n+1} = \lim_{n\to\infty}\frac{n^2(4+6/n+2/n^2)}{n^2(1+2/n+1/n^2} = \lim_{n\to\infty}\frac{4+6/n+2/n^2}{1+2/n+1/n^2} = 4,$$

which is greater than 1, so by the Ratio Test the series $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}$ is divergent.

Solutions should show all of your work, not just a single final answer.

1. Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n+1}$ is absolutely convergent, conditionally convergent, or divergent. Specify clearly which convergence test(s) you use.

2. Determine if $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^3}{4^n}$ is absolutely convergent, conditionally convergent, or divergent. Specify clearly which convergence test(s) you use.

3. Determine if $\sum_{n=1}^{\infty}\frac{n^2}{n!}$ is absolutely convergent, conditionally convergent, or divergent. Specify clearly which convergence test(s) you use.

4. T/F (with justification)

   Convergence of the p-series for each $p > 1$ can be shown with the Ratio Test.

5. T/F (with justification)

   There are infinite series whose terms can be rearranged to converge to a different value.