11.10 Taylor and Maclaurin Series

Taylor series and Taylor polynomials of a function at a. If $f(x)$ can be written as a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ in an interval around a then c_n must be $\frac{f^{(n)}(a)}{n!}$. We call

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

the *Taylor series* of $f(x)$ at a and we call the partial sum

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the *nth-degree Taylor polynomial* of $f(x)$ at a. Ideally $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ for x near a, or equivalently $f(x) = \lim_{n\to\infty} T_n(x)$ for x near a, and to verify this in examples we can use Taylor's inequality below.

Maclaurin series. The Taylor series of $f(x)$ at $a=0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

and is called the *Maclaurin series* of $f(x)$[1]. Ideally $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ for x near 0.

Taylor's inequality. A bound on the remainder $R_n(x) = f(x) - T_n(x)$, where $T_n(x)$ is a Taylor polynomial for $f(x)$ at a, is *Taylor's inequality*, which uses a bound on $|f^{(n+1)}(x)|$:

$$\text{if } |f^{(n+1)}(x)| \le M \textbf{ for all } |x-a| \le d \text{ then } |R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \text{ if } |x-a| \le d.$$

Important Maclaurin series representations.

Function	Validity	Function	Validity
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$-1 < x < 1$	$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	all x
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	all x	$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	all x
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	$-1 < x \le 1$	$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 \le x \le 1$

[1] The term "Maclaurin series" has a peculiar status: it essentially exists only in calculus courses. People who use power series regularly, in math or physics, speak instead about a Taylor series or power series at 0.

Example: Compute the Taylor series for $f(x) = \ln(x)$ at $a = 10$ and use Taylor's inequality to show when $|x - 10| \le 4$ that $|R_n(x)| = |\ln(x) - T_n(x)| \to 0$ as $n \to \infty$.

Thinking about the problem:

We will differentiate $\ln x$ enough times to see a pattern. The pattern will give us the coefficients in the Taylor series and help us bound $|f^{(n+1)}(x)|$ to find M in Taylor's inequality.

Doing the problem:

The first several higher derivatives of $f(x) = \ln x$ are in the table below.

n	0	1	2	3	4	5	6	7
$f^{(n)}(x)$	$\ln x$	$1/x$	$-1/x^2$	$2/x^3$	$-6/x^4$	$24/x^5$	$-120/x^6$	$720/x^7$

The pattern for $n \ge 1$ is $f^{(n)}(x) = (-1)^{n-1}\dfrac{(n-1)!}{x^n}$, so the Taylor series of $\ln x$ at $a = 10$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f^{(n)}(10)}{n!}(x-10)^n &= f(10) + \sum_{n=1}^{\infty} \frac{f^{(n)}(10)}{n!}(x-10)^n \\
&= \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{10^n n!}(x-10)^n \\
&= \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-10)^n}{10^n n} \\
&= \ln 10 + \frac{x-10}{10} - \frac{(x-10)^2}{200} + \frac{(x-10)^3}{3000} - \frac{(x-10)^4}{40000} + \cdots
\end{aligned}$$

Now we seek an M so that $|f^{(n+1)}(x)| \le M$ when $|x - 10| \le 4$, which means $6 \le x \le 14$. Since $f^{(n+1)}(x) = (-1)^n n!/x^{n+1}$, we need $|n!/x^{n+1}| \le M$ for $6 \le x \le 14$. The biggest value of $|n!/x^{n+1}| = n!/x^{n+1}$ in that x-range is $n!/6^{n+1}$, so use $\boxed{M = n!/6^{n+1}}$: if $|x - 10| \le 4$ then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-10|^{n+1} = \frac{n!/6^{n+1}}{(n+1)!}|x-10|^{n+1} = \frac{1}{n+1}\left(\frac{|x-10|}{6}\right)^{n+1} \le \frac{(2/3)^{n+1}}{n+1}.$$

Thus $R_n(x) \to 0$ as $n \to \infty$, so for $|x - 10| \le 4$, $\ln x$ equals its Taylor series at $a = 10$.

Solutions should show all of your work, not just a single final answer.

1. Let $f(x) = \sqrt{x}$.

   (a) Why does $f(x)$ not have a Maclaurin series?

   (b) Determine the 3rd-degree Taylor polynomial $T_3(x)$ for $f(x) = \sqrt{x}$ at $a = 9$. Start off by filling in the following table of higher derivatives for $f(x)$.

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0		
1		
2		
3		

   (c) Compute $T_3(10)$ from (b) to 6 digits after the decimal point. (This is an estimate for $\sqrt{10}$.)

   (d) Use Taylor's inequality to bound the error $|\sqrt{10} - T_3(10)|$.

2. Use the Maclaurin series for e^x and $\arctan x$ to find the Maclaurin series for the following functions. Determine the radius of convergence in each case.

   (a) $f(x) = e^{3x} + e^{-3x}$

   (b) $f(x) = \arctan(x/3)$

3. T/F (with justification)

   If $f(x) = 1 + 3x - 2x^2 + 5x^3 + \cdots$ for $|x| < 1$ then $f'''(0) = 30$.