Section 5.3: The Fundamental Theorem of Calculus

(1) In this section, we learn the fundamental theorem of calculus. What does the first fundamental theorem of calculus say?

> The first fundamental theorem asserts that, if f is continuous on $[a, b]$, then the function g defined by
> $$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$
> is continuous on $[a, b]$, differentiable on (a, b), and $g'(x) = f(x)$.

(2) In the first fundamental theorem of calculus, we deal with a function of the form $g(x) = \int_a^x f(t)\,dt$. What happens when you plug in different values of x into this function. What happens when we find $\dfrac{\Delta g}{\Delta x}$? Explain in terms of FTC 1.

> Plugging in different values of x results in integrating f over different intervals $[a, x]$. If x, x' are two numbers in $[a, b]$ with $x' < x$, then
> $$\frac{\Delta g}{\Delta x} = \frac{g(x) - g(x')}{x - x'} = \frac{\int_{x'}^x f(t)dt}{x - x'}$$
> is just the average of f over the interval $[x', x]$

(3) In what scenarios do we use FTC1? How does the chain rule come in to play?

> FTC1 is used in finding the derivative of an integral defined function. If $g(x) = \int_a^x f(t)dt$, and $h(x)$ is another function, then, when the chain rule applies, one has
> $$\frac{d}{dx}\int_a^{h(x)} f(t)dt = \frac{d}{dx}g(h(x)) = h'(x)g'(h(x)) = h'(x)f(h(x)).$$

(4) The much more commonly used Fundamental Theorem of Calculus is FTC 2. It tells us how to use antiderivatives to calculate exact areas under curve. Write down the statement of FTC 2 and then explain in your own words. Explain exactly what steps you need to do to find the area under a curve.

If f is continuous on $[a, b]$, then
$$\int_a^b f(x)dx = F(b) - F(a),$$
where F is any antiderivative of f; i.e., $F' = f$.
Suppose f is a non-negative function. To find the area underneath the curve of f over $[a, b]$, one needs only to find an antiderivative F of f, and evaluate F at the two endpoints; i.e., the area is given by
$$\int_a^b f(x)dx = F(b) - F(a).$$

(5) Use our understanding of getting displacement from velocity in various ways to explain why FTC 2 makes sense.

Let v be velocity, and s position. The total displacement of the object over the time interval $[a, b]$ should be $s(b) - s(a)$. As the integral of v over $[a, b]$ is the limit of Riemann sums of v, whose summands take the form $v(t)[t_{i+1} - t_i]$, we see that the integral is effectively a sum of infinitesimal displacements. Hence, in the limit, this sum should approach the total distance traveled, i.e., $s(b) - s(a)$. This is what FTC2 asserts.

(6) Explain how you can use FTC 2 to explain FTC 1. (In fact, to prove FTC 2 the standard method uses FTC 1, which is why they are ordered the way they are.)

Choose an antiderivative F of f so that $F(a) = 0$. To see how this is done, let G be any antiderivative of f. Now let $F = G - G(a)$, and observe $F' = G' = f$, so that F is an antiderivative of f with $F(a) = G(a) - G(a) = 0$. Then FTC2 asserts
$$F(b) = \int_a^b f(t)dt,$$
with $F' = f$. This is FTC1.

Another way to think about it is:
$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}F(t)|_a^x \frac{d}{dx}(F(x) - F(a)) = F'(x) = f(x)$$
where F is an antiderivative of f and thus $F'(x) = f(x)$.

Extra Practice in Book: 5.3: 3, 5, 9, 13, 21,23, 27, 35, 37, 47, 55, 60, 64, 66