Section 4.2: The Mean Value Theorem

(1) In this section, we see Rolle's Theorem and the Mean Value Theorem. What does each theorem say? What is the connection between the two theorems?

The Rolle's Theorem:
Assume f is a function that is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then there is a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem:
Assume f is a function that is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c in (a, b) such that $f'(c)$ is the average slope of the function on (a, b), i.e.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

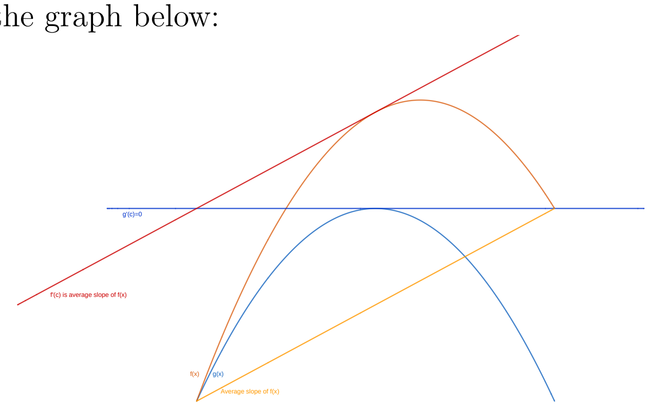
Connection:
We can easily read from the above equation that if $f(a) = f(b)$ then $f'(c) = 0$, so Rolle's Theorem can be considered as a special case of Mean Value Theorem. It turns out that we can also obtain the Mean Value Theorem from Rolle's Theorem. This description is below. You don't need to know it but it's pretty nifty! If we put

$$g(x) = f(x) - \frac{x - a}{b - a}(f(b) - f(a))$$

then $g(a) = g(b) = f(a)$ and the conclusion of Rolle's theorem applying to g (i.e. $g'(c) = 0$) would lead to the formula

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

So Mean Value Theorem can be deduced from Rolle's Theorem as well. See the graph below:



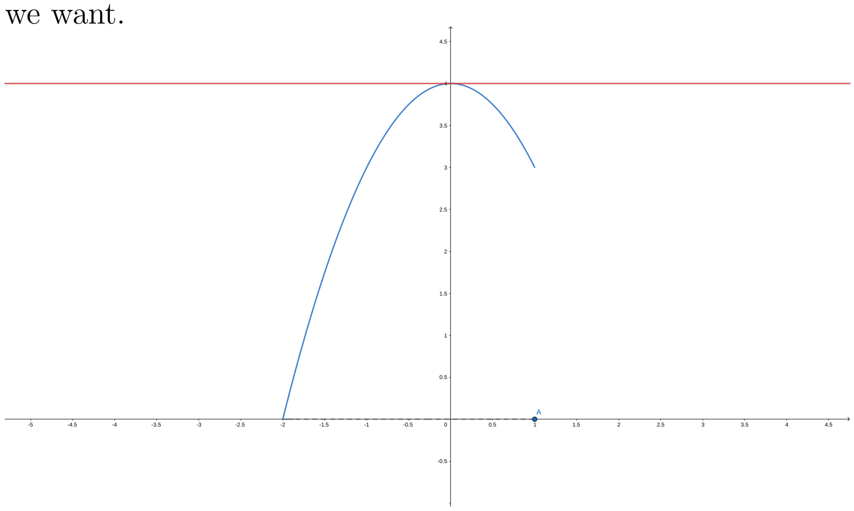
(2) In order for the Mean Value Theorem to hold, you need a function which is continuous on $[a, b]$ and differentiable on (a, b). Give examples for each of the following situations. (Sketch a graph of such a function and explain why it fits the situation.)

(a) A function which is not continuous on $[a, b]$ but is differentiable on (a, b) where the conclusion of the MVT holds.

The function

$$f(x) = \begin{cases} -x^2 + 4, & \text{if } -2 \leq x < 1 \\ 0, & \text{if } x = 1 \end{cases}$$

does the job. From the graph below, we can see that $c = 0$ is the point that we want.

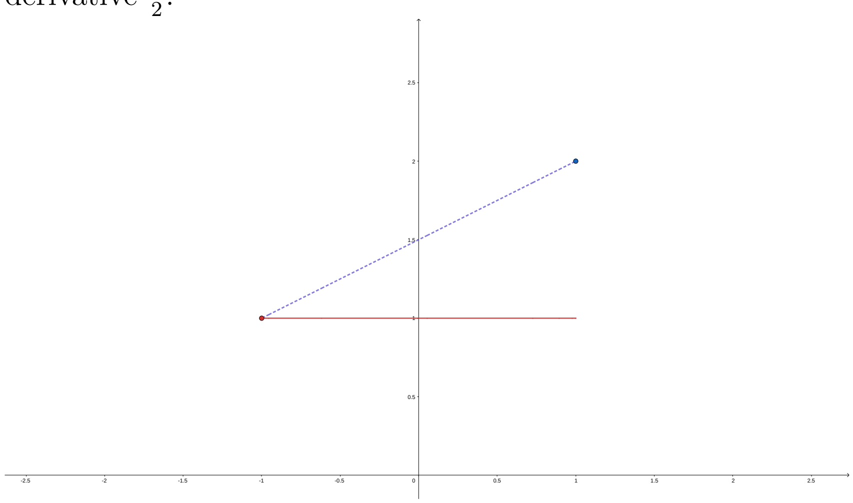


(b) A function which is not continuous on $[a, b]$ but is differentiable on (a, b) where the conclusion of the MVT does not hold.

The function

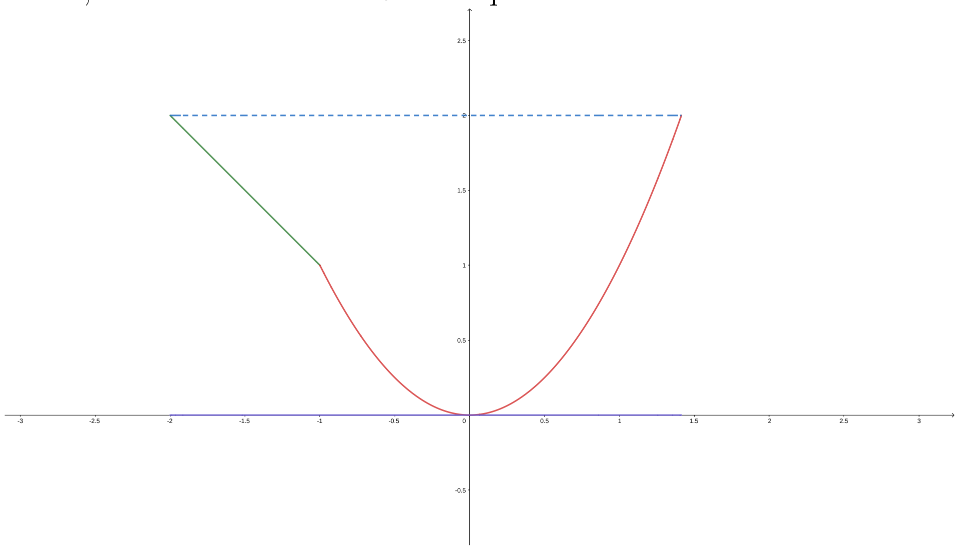
$$f(x) = \begin{cases} 1, & \text{if } -1 \leq x < 1 \\ 2, & \text{if } x = 1 \end{cases}$$

does the job. From the graph below, we can see that no point can have derivative $\frac{1}{2}$.



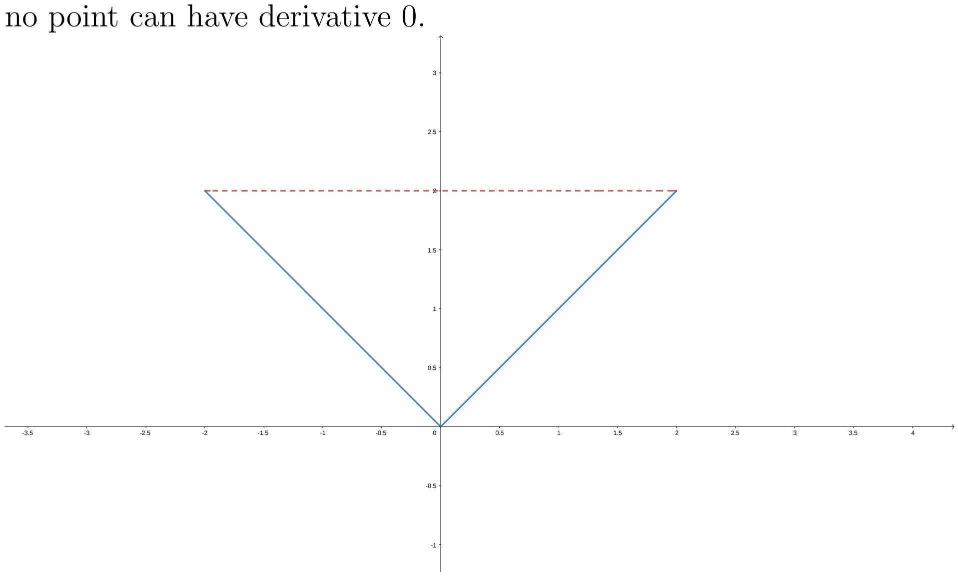
(c) A function which is continuous on $[a, b]$ but is not differentiable on (a, b) where the conclusion of the MVT holds.

The function $f(x) = \begin{cases} -x, & \text{if } -2 \leq x < -1 \\ x^2, & \text{if } -1 \leq x \leq \sqrt{2} \end{cases}$ does the job. From the graph below, we can see that $c = 0$ is the point that we want.



(d) A function which is continuous on $[a, b]$ but is not differentiable on (a, b) where the conclusion of the MVT does not hold.

The function $f(x) = |x|$ does the job. From the graph below, we can see that no point can have derivative 0.



(3) Explain how we can use the Mean Value Theorem to prove that if the derivative of a function is 0 then that function must be constant.

Suppose not the function $f(x)$ is not constant, then there are two different points a and b such that $f(a) \neq f(b)$. Since f is differentiable, we can apply the MVT so that there will be a point c in between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \neq 0,$$

which contradicts the fact that the derivative of $f(x)$ is 0.

(4) Explain how the Mean Value Theorem can be used in issuing speeding tickets

Suppose that we have two detectors in two different positions, then we can calculate the average speed of each car between those two points. If the average speed of a car exceed the speed limit during the interval, then by MVT there must be a point when this car is speeding. Note that in some sense VASCAR is based on this method. Also some countries like UK and China do have systems based entirely on this method.

(5) Compare and contrast the Mean Value Theorem and the Intermediate Value Theorem.

MVT tells the property of possible derivative of some differentiable functions while IVT describe the possible values that a continuous function can reach. However, assume the derivative of the function mentioned in MVT is continuous, then MVT can be viewed as a special case of IVT. On the other hand, MVT tells nothing about the values that original function can take.

Extra Practice in Book: 4.2: 3, 4, 7, 11, 17, 25, 27, 36,