Section 3.6: Derivatives of Logarithmic Functions

(1) In this section, implicit differentiation allows us to find the derivative $\frac{dy}{dx}$ of $y = \log_b(x)$. Reprove that
$$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}.$$

Proof. Let $y = \log_b x$. Then
$$b^y = x.$$
Differentiating this equation implicitly with respect to x, we get
$$b^y \ln b \frac{dy}{dx} = 1,$$
where we used $\dfrac{d}{dx}(b^x) = b^x \ln b$. So we have
$$\frac{dy}{dx} = \frac{1}{b^y \ln b} = \frac{1}{x\ln b}.$$
□

(2) We could have proved the same fact using the change of base formula for logs. Write out that proof as well.

Proof.
$$\frac{d}{dx}\log_b x = \frac{d}{dx}\frac{\ln x}{\ln b} = \frac{1}{\ln b}\frac{d}{dx}\ln(x) = \frac{1}{x\ln b}.$$
□

(3) Find the derivative of $y = \ln\left(\frac{x^2+1}{\sqrt{x-1}}\right)$, using the chain and quotient rules. Then redo it by first using logarithmic properties first before taking the derivative.

Using the chain rule and quotient rule, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\sqrt{x-1}}{x^2+1} \cdot \frac{2x\sqrt{x-1} - (x^2+1)/(2\sqrt{x-1})}{x-1} \\
&= \frac{1}{x^2+1} \cdot \frac{2x\sqrt{x-1} - (x^2+1)/(2\sqrt{x-1})}{\sqrt{x-1}} \\
&= \frac{1}{x^2+1} \cdot \left(2x - \frac{x^2+1}{2(x-1)}\right) \\
&= \frac{1}{x^2+1} \cdot \left(\frac{4x(x-1)}{2(x-1)} - \frac{x^2+1}{2(x-1)}\right) \\
&= \frac{1}{x^2+1} \cdot \left(\frac{4x^2-4x-x^2-1}{2(x-1)}\right) \\
&= \frac{3x^2-4x-1}{2(x-1)(x^2+1)}.
\end{aligned}$$

Now using logarithmic properties before taking the derivative, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\ln\left(\frac{x^2+1}{\sqrt{x-1}}\right) \\
&= \frac{d}{dy}\left(\ln(x^2+1) - \ln(\sqrt{x-1})\right) \\
&= \frac{d}{dy}\left(\ln(x^2+1) - (1/2)\ln(x-1)\right) \\
&= \frac{2x}{x^2+1} - \frac{1}{2(x-1)} \\
&= \frac{4x(x-1)}{2(x^2+1)(x-1)} - \frac{(x^2+1)}{2(x^2+1)(x-1)} \\
&= \frac{4x^2-4x-x^2-1}{2(x^2+1)(x-1)} \\
&= \frac{3x^2-4x-1}{2(x^2+1)(x-1)}.
\end{aligned}$$

(4) We can use logarithmic properties to help us simplify expressions before taking the derivative. Write out the important logarithmic properties.

If x and y are positive numbers, then

(a) $\log_b(xy) = \log_b x + \log_b y$

(b) $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

(c) $\log_b(x^r) = r\log_b x$ (where r is any real number).

(5) Explain how to do logarithmic properties to simplfy the process of taking the derivative of

$$f(x) = \frac{(x+1)^2(x+2)}{\sqrt{x-1}}.$$

Taking the natural log on both sides, we get

$$\begin{aligned}
\ln f(x) &= \ln\left(\frac{(x+1)^2(x+2)}{\sqrt{x-1}}\right) \\
&= \ln((x+1)^2) + \ln(x+2) - \ln((x-1)^{1/2}) \\
&= 2\ln(x+1) + \ln(x+2) - \frac{1}{2}\ln(x-1).
\end{aligned}$$

Differentiating this equation implicitly with respect to x, we get

$$\frac{f'(x)}{f(x)} = \frac{2}{x+1} + \frac{1}{x+2} - \frac{1}{2(x-1)}.$$

Thus, we get

$$\begin{aligned}
f'(x) &= f(x)\left(\frac{2}{x+1} + \frac{1}{x+2} - \frac{1}{2(x-1)}\right) \\
&= \frac{(x+1)^2(x+2)}{\sqrt{x-1}}\left(\frac{2}{x+1} + \frac{1}{x+2} - \frac{1}{2(x-1)}\right).
\end{aligned}$$

(6) (IMPORTANT!) There are a few concepts/techniques that come up multiple times in the course. One of them is the following: "If you have an x or function of x in the exponent that you need to "deal with," take the ln of both sides. This will allow us bring down the function of x". In this section, we see this technique used for taking derivatives. Explain how it works and use it for taking the derivative of $y = (\sin x)^{2\ln(x)}$.

Taking the natural log on both sides, we get

$$\ln y = \ln((\sin x)^{2\ln x}) = 2\ln x \ln(\sin x).$$

Differentiating this equation implicitly with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = 2\left(\frac{\ln(\sin x)}{x} + \ln x \cdot \frac{\cos x}{\sin x}\right).$$

Thus, we get

$$\begin{aligned}\frac{dy}{dx} &= y \cdot 2\left(\frac{\ln(\sin x)}{x} + \ln x \cdot \frac{\cos x}{\sin x}\right)\\ &= 2(\sin x)^{2\ln x}\left(\frac{\ln(\sin x)}{x} + \ln x \cot x\right).\end{aligned}$$

Extra Practice in Book: 3.6: Derivative Rules (2-30, 39-50) until comfortable with all rules. 33, 52, 53,