Section 3.5: Implicit Differentiation

(1) In this section, we learn about implicit differentiation which allows us to find the derivative $\frac{dy}{dx}$ of a function $y = f(x)$ without having to solve for y. It even works when y is not a "function" of x. Explain how implicit differentiation is related to the chain rule.

Answer.
We treat y as an implicit function of x, so when we differentiate, we take the derivative of y as the "outer" function in the chain rule and multiply it by $\frac{dy}{dx}$ (the derivative of y), which represents the "inner" function in the chain rule since the derivative of y is really the derivative of a function of x.

(2) When doing implicit differentiation, when do we get a $\frac{dy}{dx}$ term?

Answer.
Because y is a function of x, we use $\frac{dy}{dx}$ when taking the derivative of y. This is the same as when we see $\frac{d}{dx}[y]$, meaning the derivative with respect to x of a function y.

(3) When we solve for $\frac{dy}{dx}$ at a certain point, what does that value tell us?

Answer.
As stated in the answer to question (2), $\frac{dy}{dx}$ means the derivative of a function y with respect to x, so when we solve for $\frac{dy}{dx}$ at a certain point, it means we have solved for the derivative of the function at that point, i.e. we have found the slope of the tangent line at that point.

(4) Explain how to find $\sin(\arccos(x))$.

Answer.

To find $\sin(\arccos(x))$, we can start with considering the inside function, let's call it θ (since it's an angle). So we have:

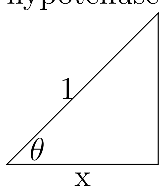
$$\theta = \arccos(x)$$

By the definition of arccos this means that θ is an angle such that

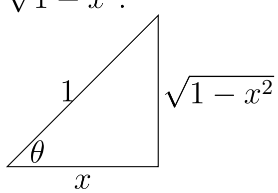
$$\cos(\theta) = x.$$

We want to find $\sin(\theta)$.

Method 1: Let's draw a triangle with an angle θ whose cos is x. Since we know $\cos(x)$ is adjacent over hypotenuse, let the adjacent side be x and the hypotenuse be 1.



Now, we want to find $\sin(\arccos(x))$ so we want to find $\sin(\theta)$. Using pythagorean theorem, we can find that the opposite side must have length $\sqrt{1-x^2}$.



Thus, using the fact that $\sin(\theta)$ can be found doing opposite over hypotenuse, we find that

$$\sin(\theta) = \sin(\arccos(x)) = \sqrt{1-x^2}$$

Method 2: Again, we let $\cos(\theta) = x$ and we want to find $\sin(\theta)$. We also know that

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

so plugging in the fact that $\cos(\theta) = x$, we get that

$$\sin^2(\theta) + x^2 = 1,$$

which means that

$$\sin^2(\theta) = 1 - x^2$$

or

$$\sin(\theta) = \sqrt{1-x^2}.$$

(5) We can use implicit differentiation to find the derivative of the inverse trig functions. Find $\frac{d}{dx}\arctan(x)$.

Answer.
Let $y = \arctan(x)$, then $\tan(y) = x$ by the definition of the arctangent function. We can now use substitution:

$$\frac{d}{dx}[y = \arctan(x)] = \frac{d}{dx}[\tan(y) = x].$$

Now take the derivative using implicit differentiation to get

$$\frac{d}{dx}[\tan(y) = x] \implies \sec^2(y) \cdot \frac{dy}{dx} = 1$$

because on the left-hand side the derivative of $\tan(x)$ is $\sec^2(x)$, which was the "outer" function of the chain rule, and the derivative of the "inner" function, y, is $\frac{dy}{dx}$. On the right hand side, the derivative of x is 1. At this point, we can solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)}.$$

The previous step is possible because from the beginning, y is the range of arctangent ($y = \arctan(x)$), so $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$, and $sec^2(x)$ is always positive in this interval. We've solved for $\frac{dy}{dx}$ on the left-hand side at this point, but we still have y on the right-hand side when we want a function of x. Recall that we can get x from $x = \tan(y)$, and by the Pythagorean Identity, $\sec^2(x) = 1 + \tan^2(x)$. Let us use that information to complete the problem:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sec^2(y)}\\ &= \frac{1}{1 + tan^2(y)}\\ &= \frac{1}{1 + x^2}.\end{aligned}$$

Therefore, $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$.

Extra Practice in Book: 3.5: Derivative Rules (5-20, 49-60) until comfortable with all rules. 27, 31, 35,