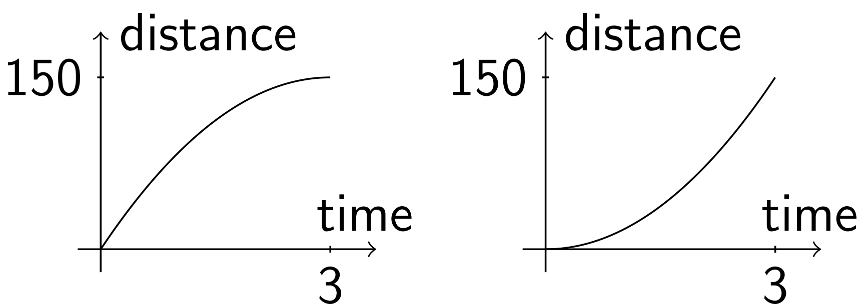
2.1 Tangent and Velocity Problems

Calculus is all about how functions describe change!

e.g. A car travels 150 miles in 3 hours. What was the car's average speed? (Speed is the rate of change of distance with respect to time).

But the car's speed could have been changing throughout the journey. Here are some possible graphs of the car's distance traveled against time.



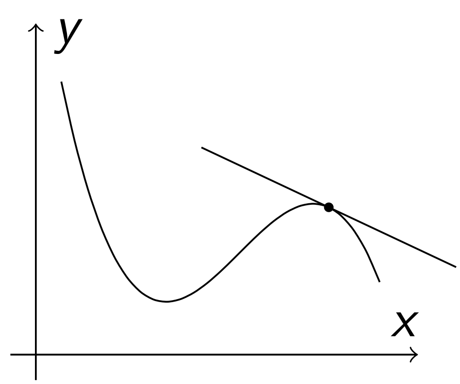
Which of these graphs represents the car speeding up over time?

(A) The graph on the left
(B) The graph on the right

What would the graph look like if the car's speed was constant throughout the journey?

The rate of change of a function corresponds to the slope of its graph. Can we find the slope of a curve?

A **tangent line** to a curve is a line that just touches (but does not cross) the curve at a point:



The slope of a curve at a particular point is the slope of the tangent line to the curve at that point.

How can we find the slope of a tangent line?

e.g. An object is dropped from 100 ft. Its height after t seconds is given by $h(t) = 100 - 16t^2$. Find the average velocity of the object over the time intervals $[1, 2]$, $[1, 1.1]$, and $[1, 1.01]$ and estimate the object's velocity at $t = 1$.

A **secant line** is a line that crosses a curve at two points. When we find the average velocity of the object from $t = 1$ to $t = 2$, we are finding the slope of the secant line that passes through those two points on the graph of $h(t)$.

Average velocity over $[1, 2]$ (i.e. slope of secant line)

$$\frac{\text{change in } h}{\text{change in } t} = \frac{\Delta h}{\Delta t} = \frac{h(2) - h(1)}{2 - 1}$$
$$= \frac{(100 - 16(2)^2) - (100 - 16(1)^2)}{1}$$
$$= (100 - 64) - (100 - 16)$$
$$= -48 \text{ ft/s}$$

Moving the right endpoint ($t = 2$) closer to the left endpoint ($t = 1$) will give us a better approximation to the tangent line!

Average velocity over $[1, 1.1]$

How can we find the slope of a tangent line?

Average velocity over $[1, 1.01]$

Can we find the equation of the tangent line to the graph of $h(t)$ at $t = 1$?

What is a reasonable estimate for the slope of the tangent line (the instantaneous velocity) at $t = 1$?

A little more practice

e.g. The point $P(-2, 1)$ lies on $y = \dfrac{5}{x^2+1}$. Estimate the slope of the curve at P by finding the slope m_{PQ} of the secant line through the points P and Q when Q is the point on the curve with x-coordinate

(a) -2.1
(b) -2.01
(c) -2.001
(d) -1.9
(e) -1.99
(f) -1.999

(Round to four decimal places).

(a) $\dfrac{\Delta y}{\Delta x} = \dfrac{y(-2.1) - y(-2)}{-2.1 - (-2)}$

$$= \frac{\frac{5}{(-2.1)^2+1} - 1}{-0.1} = 0.7579$$