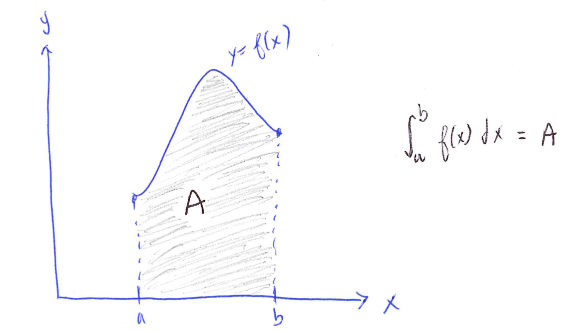
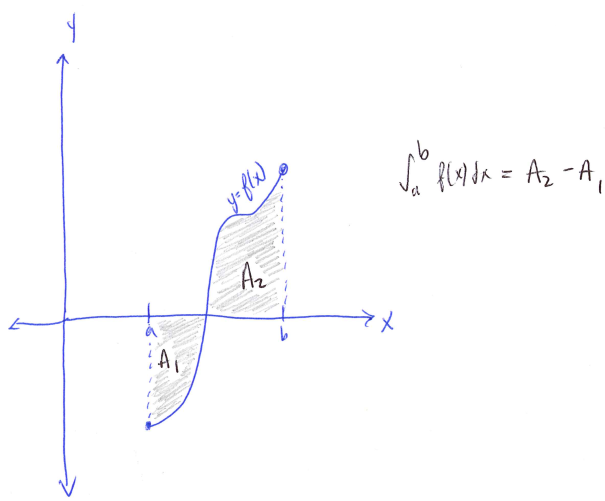
Section 5.2: Areas and Distances

(1) In this section, we expand upon our understanding of finding areas under curves. How do we express the definite integral from $x = a$ to $x = b$ of a function $f(x)$? What is the geometric interpretation of this if $f(x) > 0$, and if $f(x)$ takes on both positive and negative values.

For a function f defined over the interval $[a, b]$, the definite integral of $f(x)$ from $x = a$ to $x = b$ is $\int_a^b f(x)dx$. Using graphs we have:



The graph above is for the case when $f(x)$ takes only positive values.
If $f(x)$ takes on both positive and negative values then the definite integral will represent the sum of the areas of the rectangles that lie above the x-axis and the NEGATIVES of the areas of the rectangles that lie below the x-axis. This difference of areas is called the net area. Using graphs we have:



(2) How do we define the definite integral?

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \text{ where } \Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x$$

(3) Under what conditions does the definite integral exist?

The above definition holds if f is continuous on $[a,b]$ or f only has a finite number of jump discontinuities.

(4) What is the difference between displacement and distanced traveled? How can you find each from the graph of a velocity function?

Distance is the total length of the path traveled by an object. Displacement is the total change in position of the object no mater which path the object traveled. We know distance = velocity x time. Given the graph of a velocity function, we can find the distance traveled by an object by taking the (absolute) area under the curve because in Section 5.1, this is equivalent to taking the area of rectangles drawn under the curve. The height of the rectangles represents velocity and the width represents time i.e A= v x t = d. To find the displacement we take the area under the curve, taking into account the negative areas if f takes on negative values i.e we take the net area.

(5) How can we find the exact value of a definite integral from the graph of the function?

We can take the areas of rectangles drawn under the curve with width approaching 0 because that is the geometric definition of the Riemann sum in the definite integral. The smaller the width of the rectangles, the more accurate our estimate of the exact value of the definite integral will be. For some shapes, like rectangles, triangles or circles, we can use our area formula to get the exact area.

(6) Complete the statement assuming b is a number between a and c. Draw a picture to illustrate why it is true.

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^b f(x)\,dx$$

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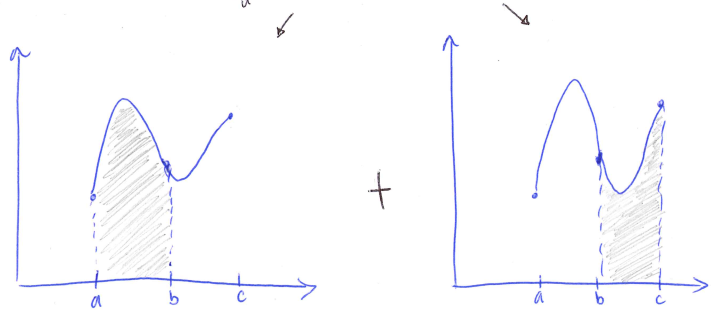

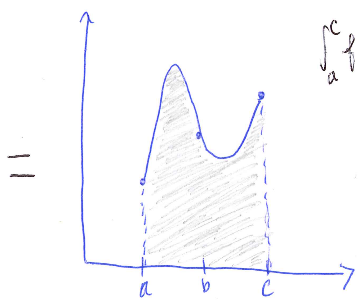

$\int_a^c f(x)\,dx$

(7) Complete each statement below, then explain geometrically in terms of area under a curve.

(a) $\int_a^b c\,dx = c(b-a)$ where c is any constant

Explanation: Since the integral $\int_a^b f(x)\,dx$ represents the area under a curve we have that $\int_a^b c$ represents the area under the constant curve c from a to b i.e. the area of a rectangle of width $(b-a)$ and length c.

(b) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Explanation: The left hand side of the equation takes the area (or net area) of the function $f(x) \pm g(x)$. While the right hand side of the equation breaks up this "whole" area and sums (or takes the difference) the area under the curve f and the area under the curve g.

(c) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ where c is any constant.

> Explanation: The integral over $cf(x)$ is the same as taking the integral of the area under the curve f shrunken or stretched by a constant c. We get the same result if we obtain the area over the curve f and then multiply by the constant c because the area is shrunken or stretched out by the same constant .

(d) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

> Explanation: The integral $\int_b^a f(x)\,dx$ takes the area under the curve from a to b (so generally speaking from left to right). So if we switched the upper and lower limits of integral we would be taking the negative of the area under the curve (because we would be going from b to a; right to left)

(e) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_b^a f(x)\,dx \geq 0$

> Explanation: $\int_b^a f(x)\,dx$ is the area under the curve f from a to b. This area is positive since the function is positive.

(f) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

> Explanation: The larger function f will have an area under the curve bigger than the area under the curve of the smaller function g

(g) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

> Explanation: If f is a positive function, this inequality states that the area under the curve f will be larger than the area of the rectangle with width $b-a$ and height m but smaller than the area of the rectangle with width $b-a$ and height M.

Extra Practice in Book: 5.2: 3, 9, 33, 37, 41, 47, 49, 51, 53, 57