Section 2.8: The Derivative as a Function

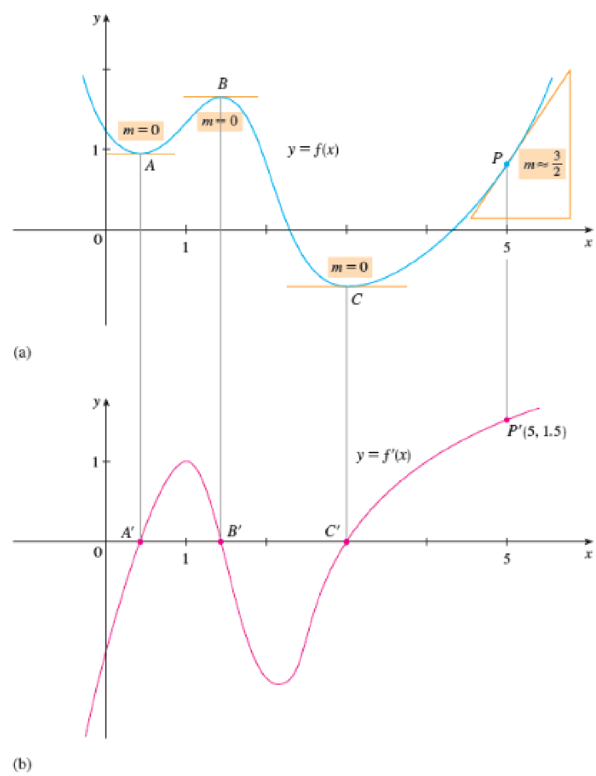
(1) In this section, we think about the derivative as a function. When we evaluate the derivative at a specific point, it tells us the slope of the tangent line to the graph of the function at that point. Write down the limit definition of derivative for a function f.

f'(x)= $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

Geometrically, $f'(x)$ can be understood as the slope of the tangent line to the graph of f at the point $(x, f(x))$. The domain of f' is the set of $x-$values such that $f'(x)$ exists. Note that the domain of f' may be smaller than the domain of f.

(2) Sketch the graph of a function with multiple peaks and valleys. By considering the slope of the tangent lines at various points, sketch a graph of the derivative function.

This graph comes from the book on page 153.



(3) We can use the limit definition of derivative to find the derivative function. This will lead to using the algebraic limit laws in the last section. Some functions you should be able to do are any linear function, quadratic or cubic, square root functions and functions of the form constant over linear term. Pick one or two of these and try them.

Example from page 154. If $f(x) = x^3 - x$, find the formula for $f'(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^3-(x+h)]-[x^3-x]}{h}$$

$$= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x-h-x^3+x}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3-h}{h}$$

$$= \lim_{h\to 0} (3x^2+3xh+h^2-1) = 3x^2-1$$

(4) What are the other notations we can use for derivatives? When might we want to use them?

$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$

D and d/dx are called **differentiable operators**. They indicate the process of differentiation.
dy/dx is the same as saying f'(x). It is useful when used with increment notation. Using Leibniz notation:
$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$

(5) What does it mean for a function to be differentiable at a point? On an interval? What are four different things that can occur in a graph that lead to the function not being differentiable at that point?

A function is **differentiable at a point** a if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$) if it is differentiable at every number in the interval.

Four different things that can occur in a graph that lead to a function not being differentiable at a point are:

(a) corner
(b) cusp (like a corner but derivatives from both sides are approaching $\pm\infty$
(c) discontinuity
(d) vertical tangent line when $x = a$; that is, f is continuous at a and $\lim_{x\to a} |f'(x)|=\infty$
(e) Note: Another way to look at differentiability is from a graphing calculator. If the function is differentiable at a point, we should be able to zoom in toward the point and the graph will look more like a straight line.

(6) Is it possible for a function to be continuous but not differentiable? Differentiable but not continuous? If yes, given an example.

A continuous function does not need to be differentiable. For example, the absolute value function is continuous(though not differentiable at x=0).
A differentiable function must be continuous

(7) What does it mean to take higher derivatives? If you function is position, what does the second derivative tell us? The third derivative?

If f is a differentiable function, then its derivative f' is also a function, f' may have a derivative of its own, denoted by $(f')' = f''$. This new function is called the second derivative of f because it is the derivative of the derivative of f. Differentiating can continue to find the third, fourth, and successive derivatives of $f(x)$. This is called higher order derivatives of $f(x)$. If the function is position, the second derivative tells us acceleration $a(t)$ and the third derivative tells us jerk. Jerk is the rate of change of acceleration.

Formulas/Ideas to Know

$$
\begin{aligned}
\text{Slope of the tangent line at } (x) &= \text{derivative at } x \\
&= f'(x) \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \text{instantaneous rate of change at } x \\
&= \text{instantaneous velocity function (if } f \text{ is a position function)}
\end{aligned}
$$

Extra Practice in Book: 2.8: 1, 3, 5, 17, 19, 21, 25, 27, 29, 41 ,43, 51,